Occupation bounds for coupled permanent cofactors

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Luminary Oddities

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Abstract

For a complex positive semidefinite Gram matrix $B = VV^*$, we couple the permanent-cofactor contractions of B and $|B|^2$ and prove the sharp all-order bound $0 \preceq U - aT/n! \preceq (n-1)(c - a^2/n!)I$, where $a = \operatorname{per} B$ and $c = \operatorname{per}|B|^2$. For the actual bordered permanent defect, we identify a nonnegative cofactor-weighted variance E and characterize its zero set. For arbitrary Gram rows in every order, we prove $E \geq (\alpha_n/n!)(na - t)$, where α_n is the top ordinary selected-mode occupation coefficient and t its unnormalized first moment. The universal prefactor is sharp. An occupation-resolved refinement links the actual scaled variance coefficients to lower ordinary occupations. Orthogonal trace sectors give a joint occupation and frame-variance bound. For six unit Gram factors, $E \geq v^2/270 + 7\mathcal{A}/1620$; at rank at most five, $E \geq 6(v^2 + \mathcal{A})/625$, where v is the occupation variance and $\mathcal{A}$ the variance of the frame operator. An all-order trace-one positive-operator formulation includes mixed factors with their actual operator variance. The quadratic power is sharp. The proofs allow arbitrary complex phases, singular limits and kernel mixtures. We give quantitative base-defect reserves, retain the inherited smaller-order defect, and prove uniform sufficient compensation domains by rank-one secants with a retained top occupation coefficient. Frame-variance floors strengthen exact occupation-box matrix certificates. Retaining the transformed defect multiplicatively yields further conditional compensation domains. A finite-forward identity isolates a nonnegative curvature remainder and the degree-six ordinary occupation loss. An explicit rank-one pencil rules out uniform positive multiplicative transfer of the base defect to a prescribed forward point. Relative Gram-factor stability, occupied-vertex domains, repeated-block neighborhoods and the coherence-$1/15$ certificate are retained. A self-contained reconstruction of the attributed small-order induction supplies scalar inequalities through order six. The trace and occupation reserves are combined with their exact projection overlap, and actual cofactor occupations give the sharp curvature floor $R_{\sqrt{2}-1} \geq (19/240)\alpha_6^2$, sharp also on active ranks five and six. A common-factor coefficient argument proves compensation whenever one of six selected occupations is at most $1/31$. Consequently every complex PSD matrix of order seven with one normalized squared off-diagonal modulus at most $1/31$ satisfies the self-conjugate Chollet inequality; the two-factor conclusion requires both factors to meet that condition. The remaining positive-loss residual chart has $K_{ii} < 30$. Classical weighted permanent bounds and balancing yield further explicit coherent-column and spectral covariance-deficit sufficient domains. The original residual occupation polynomial, thinning law and nonreal-root examples are retained.

1 Introduction and notation

For a square matrix A, $\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_i A_{i,\sigma(i)}$; $A(j \mid i)$ deletes row j and column i. The symbol $\circ$ denotes entrywise multiplication, $*$ the conjugate transpose, and $|B|^2 = B \circ \overline{B}$, not the operator square.

Throughout, $V \in \mathbb{C}^{n \times d}$, $B = VV^*$, and d is arbitrary. Define

$$a = \operatorname{per} B, \qquad c = \operatorname{per}|B|^2, \qquad R_{ij} = \operatorname{per} B(j \mid i), \qquad L_{ij} = \operatorname{per}|B|^2(j \mid i), \tag{1}$$

$$T = V^* R V, \qquad U = V^*(L \circ B)V, \qquad r = c - \frac{a^2}{n!}, \qquad S = U - \frac{a}{n!}T. \tag{2}$$

All operator inequalities below act on $\mathbb{C}^d$ unless specified otherwise. No rank, normalization of the diagonal, or invertibility hypothesis is imposed.

Permanent-cofactor Gram matrices and their Schur products are classical. Paksoy [4], in particular, uses omitted symmetric tensors and a weighted cofactor matrix with constant permanent row sum. Symmetrizers and bosonic permanent states are standard [5, 6]; Schur weights and content eigenvalues are standard representation theory [7, 8]. Our focus is the *joint residual S*, its sharp norm bound, and a computable refinement that preserves its common Gram origin. Li, Zhao and Luo [3] represent coefficients of ordinary and doubled bordered permanent polynomials by squared norms over complete occupation sectors. The polynomial studied below instead comes from the joint residual projection $P_{\mathrm{diag}} - P_{\mathrm{sep}}$; its coefficients measure one shared residual state. The separate positivity of T and U, or a bound on either one alone, does not supply this residual estimate.

The motivation is the compensation in a bordered permanent inequality. Chollet asks whether $\operatorname{per}(A \circ D) \le \operatorname{per} A \operatorname{per} D$ for complex positive semidefinite A, D; see [1, 2]. The results here concern coupled cofactor operators and sufficient border certificates, rather than an unrestricted solution. They are distinct from the author's rank-restricted scalar results [11] and cofactor-comparison limitations [12]. We consolidate the sharp residual estimate and its refinements in one note.

The boundary theorem in Section 23 adds an explicit order-seven sufficient domain, including full-rank matrices. Its joint root-product proof uses common-factor coefficient constraints, weighted Cauchy–Schwarz and an elementary PSD permanent upper bound. The preceding boundary proof also uses the classical Carlen–Lieb–Loss permanent inequality [18]. The sharp retained-curvature floor and exact trace–occupation overlap retain information lost by independently summing scalar reserves. Section 24 gives further sufficient conditions by classical matrix scaling [19]. Earlier sufficient conditions, including those of Sa-nguansin and Rodtes [20], are credited.

2 A sharp bound in every order

Theorem 2.1. *With* (1)–(2)*, for every* $n \ge 2$*,*

$$0 \le S \le (n-1) r I_d. \tag{3}$$

The coefficient $n - 1$ *cannot be reduced for actual Gram factors. The conclusion includes arbitrary complex phases and singular factors.*

2.1 The two invariant projections

We use the conjugate-linear-first inner product. Regard row i of V as the ket $v_i = \sum_j V_{ij} e_j$ in $\mathcal{H} = \mathbb{C}^d$. Its physical Gram is $\overline{B}$. Put

$$f = v_1 \otimes \cdots \otimes v_n, \qquad W = f \otimes \overline{f}.$$

On $\mathcal{H}^{\otimes n} \otimes \overline{\mathcal{H}}^{\otimes n}$, let P_{sep} average independent permutations of the two legs, and let P_{diag} average equal simultaneous permutations. Both are orthogonal projections and $P_{\mathrm{sep}} P_{\mathrm{diag}} = P_{\mathrm{sep}}$. In particular $a = n! \| \operatorname{Sym} f \|^2 \ge 0$. Direct expansion gives

$$\| P_{\mathrm{sep}} W \|^2 = \frac{a^2}{(n!)^2}, \qquad \| P_{\mathrm{diag}} W \|^2 = \frac{c}{n!}. \tag{4}$$

For unit $z \in \mathbb{C}^d$, let $p_z = |\overline{z}\rangle\langle\overline{z}|$ and let $N_z = \sum_{j=1}^n p_z^{(j)}$ act on the first leg. The conjugation in p_z matches the row convention $B = VV^*$. This operator commutes with both projections.

Set $h = (P_{\mathrm{diag}} - P_{\mathrm{sep}}) W$. For $x = Vz$, differentiation of the two permanent norm formulas gives

$$\| h \|^2 = \frac{r}{n!}, \tag{5}$$

$$\langle h, N_z h \rangle = \frac{1}{n!} \left(z^* U z - \frac{a}{n!} z^* T z \right). \tag{6}$$

Here the conjugate physical Gram deformation is $B + t\,xx^*$, and

$$\frac{d}{dt}\operatorname{per}(B + t\,xx^*)\Big|_{t=0} = z^*Tz.$$

The paired deformation is $|B|^2 + t(B \circ \overline{xx^*})$, whose permanent derivative is z^*Uz. For example the latter derivative is $\sum_{ij} L_{ji}B_{ij}\overline{x_i}x_j$; L is real symmetric, yielding the defined U. The independent-projection expectation is $a(z^*Tz)/(n!)^2$ and the diagonal one is $(z^*Uz)/n!$. Their difference is (6); the cross terms vanish because N_z preserves the nested orthogonal components.

2.2 Excluding full occupation

Proof of Theorem 2.1. The spectrum of N_z is contained in $\{0, \dots, n\}$. Its n-eigenspace has first leg $\overline{z}^{\otimes n}$. A diagonal-invariant vector in that eigenspace has a symmetric first leg; diagonal invariance then forces its second leg to be symmetric too. Thus the n-occupation part of range P_{diag} lies in range P_{sep} and is absent from h. Consequently

$$0 \le \langle h, N_z h\rangle \le (n-1)\|h\|^2.$$

Use (5)–(6) and vary unit z.

For sharpness, take $n-1$ equal unit rows in one coordinate and one unit row in an orthogonal coordinate. Then $B = J_{n-1} \oplus [1]$, $a = c = (n-1)!$ and, in the repeated-row direction, $z^*Tz = (n-1)a$, $z^*Uz = (n-1)c$. The upper bound is an equality. □

In particular $r \ge 0$. If $r = 0$ then $S = 0$. Both T and U vanish on $\ker V$. Therefore, with $\mathcal{A} = (\ker V)^\perp$ and its orthogonal projection $P_{\mathcal{A}}$,

$$0 \le S \le (n-1)rP_{\mathcal{A}}. \tag{7}$$

This statement retains the kernel rather than replacing the ambient space.

3 The occupation polynomial and a cancellation

For unit z, write $x = Vz$, $X = xx^*$ and $H = B - X \ge 0$.

Proposition 3.1. *The polynomial*

$$G_z(s) = \operatorname{per}\big(|B|^2 + (s-1)(B \circ \overline{X})\big) - \frac{a}{n!}\operatorname{per}\big(B + (s-1)X\big) = \sum_{k=0}^{n} q_k(z)s^k \tag{8}$$

has $q_k(z) \ge 0$ and $q_n(z) = 0$. More precisely, if E_k is the k-occupation spectral projector of N_z, then $q_k(z) = n!\|E_k h\|^2$. Thus

$$G_z(1) = r, \qquad G_z'(1) = z^*Sz, \tag{9}$$

$$q_0(z) = \operatorname{per}(H \circ \overline{B}) - \frac{a}{n!}\operatorname{per} H. \tag{10}$$

Proof. For real $s \ge 0$ multiply the selected first-leg mode by $\sqrt{s}$. The resulting operator is $s^{N_z/2}$ and commutes with both projections. The diagonal projected squared norm is the first permanent in (8) divided by $n!$; the independent projected squared norm is $a\ \operatorname{per}(B+(s-1)X)/(n!)^2$. Their difference is $\|s^{N_z/2}h\|^2$. Polynomial identification proves the coefficient formula. The full-occupation exclusion proves $q_n = 0$. The identities at $s = 1, 0$ follow directly. □

Proposition 3.2 (Exact kernel thinning)**.** *Let $w \in \mathcal{A}$ and $\kappa \in \ker V$ be unit vectors. For $0 \le p \le 1$ and real θ put $z_p = \sqrt{p}\, w + e^{i\theta}\sqrt{1-p}\,\kappa$. Then*

$$G_{z_p}(s) = G_w(1 - p + ps), \tag{11}$$

$$q_k(z_p) = \sum_{j=k}^{n-1} \binom{j}{k} p^k (1-p)^{j-k} q_j(w). \tag{12}$$

In particular $G'_{z_p}(1) = pG'_w(1)$ and $G''_{z_p}(1) = p^2 G''_w(1)$. The formulas include the endpoints and every relative complex phase.

Proof. $Vz_p = \sqrt{p}\, Vw$, and substitution into (8) gives the first identity. Binomial expansion gives the rest. □

Proposition 3.3 (Top occupation)**.** *Put $b_i = \prod_{j \ne i} x_j$, $y = \overline{b}$, $J = \mathbf{1}\mathbf{1}^*$ and*

$$F_0 = R \circ \overline{B} - \frac{a}{n} J.$$

Then $F_0 \succeq 0$, $F_0\mathbf{1} = 0$, and

$$q_{n-1} = y^* \left(R \circ |B|^2 - \frac{a}{n} B \right) y = y^*(B \circ F_0)y = y^*(H \circ F_0)y. \tag{13}$$

These are polynomial identities, including when some $x_i = 0$.

Proof. The usual omitted symmetric-tensor Gram construction gives $R \succeq 0$. The Schur product theorem gives $F = R \circ \overline{B} \succeq 0$. Column-$i$ Laplace expansion gives $\sum_j R_{ij}\overline{B_{ij}} = a$. Thus $F\mathbf{1} = a\mathbf{1}$ and removing that Hermitian eigencomponent gives $F_0 \succeq 0$. This construction is classical; compare [4].

In powers of $t = s - 1$, the coefficient of t^{n-1} in the first permanent in (8) is $\sum_{ij} |B_{ij}|^2 \operatorname{per} B(i \mid j)\overline{b_i} b_j$. Exchanging i, j gives $y^*(R \circ |B|^2)y$. The coefficient in the second permanent is $(n-1)!\, y^* B y$. Both leading t^n terms in G_z equal $a \prod_i |x_i|^2$ and cancel. Translating $t = s - 1$ therefore leaves the next coefficient unchanged, proving the first equality in (13). Finally $\operatorname{diag}(\overline{x})y = \overline{\prod_i x_i}\,\mathbf{1}$, so $y^*(X \circ F_0)y = 0$. This proves the last equality without division. □

For later directional tests at $n = 6$, set $r_+ = r - q_0$ and $m_2 = G''_z(1)$. The elementary moment bound is

$$z^* S z \le \min \left\{ 5r_+,\ \frac{m_2}{2j} + \frac{j+1}{2} r_+ :\ j = 1, \dots, 4 \right\}. \tag{14}$$

Indeed $(k - j)(k - j - 1) \ge 0$ at integer $1 \le k \le 5$. Multiplication by q_k and summation proves each affine bound. For fixed feasible r_+, m_2, mixtures on consecutive occupations attain its segments, so this is the exact finite-support relaxation. It is an elementary instance of discrete moment linear programming [9]. Retaining q_5 also gives $z^* S z \le 4(r - q_0) + q_5$. Neither estimate alone couples its inputs to the compensation budget.

4 A central-transposition refinement at order six

In this section $n = 6$. On the first tensor leg let $C = \sum_{i<j} U_{(ij)}$ be the sum of transpositions and put

$$v = 720\, \langle h, Ch \rangle, \qquad M = \frac{v + 21r}{6}.$$

Theorem 4.1. *For every complex Gram factor of order six,*

$$0 \le S \le M I_d \le 5r I_d. \tag{15}$$

For each transposition π form $A^{\pi}_{ij} = B_{\pi(i),j}\overline{B_{ij}}$. Then

$$v = \sum_{\pi \text{ transposition}} \operatorname{per} A^{\pi} - \frac{15a^2}{720}. \tag{16}$$

The conclusion holds in every ambient dimension and at singular boundaries.

Proof. On a first-leg Schur sector of shape $\lambda \vdash 6$, the largest one-mode occupation is λ_1: column strictness in semistandard tableaux permits a fixed letter at most once in each column. The central transposition sum acts by the content sum $\kappa_\lambda = \sum_{(i,j)\in\lambda}(j-i)$; see [8, Example 3.15]. The full comparison is

λ	λ_1	κ_λ	$(\kappa_\lambda+21)/6$
(6)	6	15	6
(5, 1)	5	9	5
(4, 2)	4	5	13/3
(4, 1, 1)	4	3	4
(3, 3)	3	3	4
(3, 2, 1)	3	0	7/2
(3, 1, 1, 1)	3	−3	3
(2, 2, 2)	2	−3	3
(2, 2, 1, 1)	2	−5	8/3
(2, 1, 1, 1, 1)	2	−9	2
(1^6)	1	−15	1

so $N_z \preceq (C+21I)/6$. The separately symmetric first-leg sector is absent from h: first-leg symmetry and diagonal invariance force second-leg symmetry. All remaining sectors have $\kappa_\lambda \le 9$. Hence

$$720\,\langle h, N_z h\rangle \le (v+21r)/6 \le 5r.$$

Use (6). Nonnegativity follows from Theorem 2.1.

The class sum C commutes with both projections, and acts as 15 on their separately symmetric range. Consequently

$$\langle h, Ch\rangle = \langle W, CP_{\mathrm{diag}}W\rangle - 15\|P_{\mathrm{sep}}W\|^2.$$

Expand the diagonal average and the fifteen swaps. Summing over its permutation index gives $\operatorname{per} A^{\pi}$ for each π (equivalently its complex conjugate under the opposite permutation convention). The sum is real since CP_{diag} is self-adjoint. Equations (4) give (16). □

More precisely, if $r_\lambda = 720\|E_\lambda h\|^2$, then $S \preceq (\sum_\lambda \lambda_1 r_\lambda)I$ and $v = \sum_\lambda \kappa_\lambda r_\lambda$. The single statistic v compresses these masses. Since S vanishes on $\ker V$, (15) also gives $S \preceq MP_{\mathcal{A}}$.

5 Compensation and exact examples

For any n define $\delta = a^2 - c$ and $D_0 = \delta I_d + 2aT - U$. Let P_i be the rank-one matrix $(P_i)_{\alpha\beta} = \overline{V_{i\alpha}}V_{i\beta}$, and set

$$\mathcal{K} = \delta I_{d^2} + a(T\otimes I + I\otimes T) + T\otimes T - \sum_{ij} L_{ij}P_i\otimes P_j. \tag{17}$$

Lemma 5.1. *$D_0 \succeq 0$ implies $\mathcal{K} \succeq 0$. For the bordered Gram matrix $\widetilde{B}_z = \begin{pmatrix} B & Vz \\ z^*V^* & \|z\|^2 \end{pmatrix}$,*

$$\operatorname{per}(\widetilde{B}_z)^2 - \operatorname{per}|\widetilde{B}_z|^2 = \langle z\otimes z, \mathcal{K}(z\otimes z)\rangle.$$

Proof. L is real symmetric PSD by the omitted-tensor Gram construction. Factor $L = CC^T$ over the reals and put $A_k = \sum_i C_{ik} P_i$. Then A_k are Hermitian and $\sum_k A_k^2 = U$, since $P_i P_j$ has entries $\overline{V_{i\alpha}} B_{ij} V_{j\beta}$. Expansion gives the identity

$$\mathcal{K} = \frac{D_0 \otimes I + I \otimes D_0}{2} + T \otimes T + \frac{1}{2} \sum_k (A_k \otimes I - I \otimes A_k)^2.$$

Every displayed term is PSD when $D_0 \succeq 0$. Finally border expansion gives $\operatorname{per} \widetilde{B}_z = a\|z\|^2 + z^* T z$ and $\operatorname{per} |\widetilde{B}_z|^2 = c\|z\|^4 + \sum_{ij} L_{ij} |(Vz)_i|^2 |(Vz)_j|^2$. Subtracting proves the formula. □

Theorem 2.1 gives, in every order,

$$D_0 \succeq \left[\left(1 + \frac{n-1}{n!} \right) a^2 - nc \right] I_d + \left(2 - \frac{1}{n!} \right) aT.$$

At order six this certifies $D_0, \mathcal{K} \succeq 0$ whenever $a > 0$ and $c/a^2 \le 145/864$. Theorem 4.1 improves the sufficient test to

$$D_0 \succeq (\delta - M) I_d + \frac{1439a}{720} T, \qquad M \le \delta. \tag{18}$$

This is a condition on the actual mixed-permanent statistic.

Here are exact factors witnessing strict improvement of the scalar test. Let $W \in \mathbb{C}^{6\times 6}$ have first three rows $(1, \frac{1}{2}, 0, 0, 0, 0)$, $(1, \frac{i}{2}, 0, 0, 0, 0)$, $(1, -\frac{1}{2}, 0, 0, 0, 0)$, and last three rows e_2^T, e_3^T, e_4^T, using coordinates $0, \dots, 5$. For $0 \le i \le 5$, $0 \le j \le 4$ set

$$E_{ij} = \frac{((i+2j) \bmod 3) - 1 + i_c(((2i+j) \bmod 3) - 1)}{10000}, \qquad E_{i5} = 0,$$

where $i_c = \sqrt{-1}$ is distinguished from the row index. Put $V_5 = W + E$ and $V_6 = V_5 + 10^{-4} e_2 e_5^T$. The accompanying exact-rational verifier proves, for $B_j = V_j V_j^*$,

$$\operatorname{rank} V_j = j, \quad B_j \text{ has no zero entries}, \qquad \frac{c}{a^2} > \frac{145}{864}, \quad \frac{\delta - M}{a^2} > \frac{17}{1000} \quad (j = 5, 6).$$

Both have a triangle $B_{ij} B_{jk} B_{ki}$ with nonzero imaginary part, which cannot be removed by diagonal unitary row gauges. Numerically c/a^2 is about 0.20174. Strict margins persist on relative open neighborhoods of the corresponding rank strata. These examples distinguish two sufficient scalar certificates.

For unit z, writing $t = z^* T z$, the remaining directional target is

$$z^* S z \le \delta + \frac{1439a}{720} t + t^2.$$

Moment bounds such as (14) can certify individual directions. For a rank-five factor in six ambient coordinates, Proposition 3.2 retains every active/kernel mixture.

6 Actual residual polynomials need not be real-rooted

The positivity in Proposition 3.1 does not imply real stability. This limitation matters when importing probability-law inequalities. Consider the Gaussian-integer factor

$$V_6 = \begin{pmatrix} -1 & -i & 1-i & 1-i & -1+i & 0 \\ -1-i & 1 & i & -1 & -1 & -1 \\ i & -1-i & -1-i & 1+i & -1-i & 1-i \\ -1-i & 1+i & i & -1+i & -1 & -1-i \\ -1-i & 1-i & 1-i & -i & 1+i & -1-i \\ -1+i & -1-i & -i & 1+i & 1 & -1+i \end{pmatrix}.$$

Let V_5 be the same matrix with its last column zero, and take $z = e_0$. Here the names V_5, V_6 denote these obstruction factors only. Exact elimination gives ranks five and six; both Gram matrices have full support and nonzero imaginary triangle products. The coefficient lists $(q_0, \dots, q_5)$ are

$$\begin{aligned}\operatorname{rank} V = 6: &\quad (20542128507952/45,\ 5821468652200/9,\ 18083778219968/45,\\ &\qquad 2049107101804/15,\ 127236053248/5,\ 6211082816/3),\\ \operatorname{rank} V = 5: &\quad (403462265536/5,\ 127255137024,\ 458165154544/5,\\ &\qquad 187047753024/5,\ 8625123712,\ 878481472).\end{aligned}$$

All entries are positive and $q_6 = 0$. Nevertheless

$$4q_1^2 - 10q_0q_2 = \begin{cases} -651686468368265612895710 72/405, & \operatorname{rank} V = 6,\\ -458273045040072984596 48/5, & \operatorname{rank} V = 5.\end{cases}$$

If a degree-five polynomial with positive coefficients had only real roots, they would be negative and it could be written $q_0 \prod_{\ell=1}^{5}(1 + \rho_\ell s)$ with $\rho_\ell > 0$. Cauchy–Schwarz then gives the necessary Newton inequality $4q_1^2 \geq 10q_0q_2$. The exact reverse inequalities prove nonreal roots. This elementary condition belongs to the classical theory of real-rooted probability generating polynomials; see [10].

In the rank-five example $e_5 \in \ker V_5$. For every $p > 0$, a nonreal root ρ is carried by Proposition 3.2 to $(\rho - 1 + p)/p$, still nonreal. Thus every nontrivial active/kernel mixture along this ray retains the obstruction, independently of relative complex phase. At $p = 0$ the polynomial is constant. These are not compensation counterexamples: the exact directional budgets $(a + t)^2 - c - z^*Uz$ are respectively 906092392444480 and 337221858201088.

7 Variance geometry of the actual border gap

In this section B is a correlation matrix. Set $w_i = V^*e_i$, so $w_i^*w_j = B_{ij}$ and $\|w_i\| = 1$. For unit z put $p_i = |w_i^*z|^2$, $P_i = w_iw_i^*$, $t = z^*Tz$ and $\delta = a^2 - c$. The normalized bordered matrix and its exact defect are

$$A_z = \begin{pmatrix} B & Vz \\ z^*V^* & 1 \end{pmatrix}, \tag{19}$$

$$\mathcal{D}(z) = (\operatorname{per} A_z)^2 - \operatorname{per}|A_z|^2 = \delta + 2at + t^2 - p^{\mathsf{T}}Lp. \tag{20}$$

Indeed, border expansion gives $\operatorname{per} A_z = a + t$ and $\operatorname{per}|A_z|^2 = c + p^{\mathsf{T}}Lp$. Thus the sufficient cofactor surrogate $Q(z) = \delta + 2at + t^2 - z^*Uz$ discards the nonnegative term

$$E(z) = z^*Uz - p^{\mathsf{T}}Lp = \sum_{i,j} L_{ij}\langle r_i, r_j\rangle, \qquad r_i = (I - zz^*)P_iz, \qquad \mathcal{D}(z) = Q(z) + E(z). \tag{21}$$

Here $U = \sum_{i,j} L_{ij}P_iP_j$. Equivalently, writing $L = CC^{\mathsf{T}}$ with real C and $H_j = \sum_i C_{ij}P_i$, $E = \sum_j [z^*H_j^2z - (z^*H_jz)^2]$. These are ordinary Hermitian variances.

Proposition 7.1 (Cofactor rank and variance rigidity)**.** *Partition the unit Gram vectors into m classes of parallel complex lines. Then* $\operatorname{rank} L = m$*, its kernel consists of the vectors whose entries sum to zero within each class, and its range consists of vectors constant within each class. If $\mu = \lambda_{\min}^{+}(L)$, then, for every unit z,*

$$E(z) \geq \mu \sum_i p_i(1 - p_i).$$

Consequently $E(z) = 0$ precisely when every w_i is parallel or orthogonal to z.

Proof. The doubled vectors $d_i = w_i \otimes \overline{w_i}$ have Gram matrix $|B|^2$. They coincide within a parallel class and are nonproportional between classes. Omitted symmetric products of the d_i have Gram matrix $L/(n-1)!$. Identify the distinct d_i with nonassociate linear forms $\ell_1, \dots, \ell_m$, of multiplicities n_j. After cancelling

$\prod_j \ell_j^{n_j-1}$, their distinct omitted products are $\prod_{h\neq j} \ell_h$. A linear relation, reduced modulo ℓ_j, has only its jth term left. Unique factorization forces that coefficient to vanish. This is the elementary reciprocal linear-form independence also underlying [14]. It proves the asserted Gram kernel and range.

Within each class P_i, and hence r_i, are identical. Each coordinate column of $(r_i)_i$ therefore lies in range L. The positive-spectrum bound for L in (21) yields $E \geq \mu \sum_i \|r_i\|^2 = \mu \sum_i p_i(1-p_i)$. It vanishes exactly when every p_i is zero or one. □

If exactly $k \geq 1$ vectors are parallel to z in this zero-variance case, row phases and ordering split $B = J_k \oplus B_\perp$. Direct block permanents give

$$t = ka, \qquad p^{\mathsf{T}} L p = kc, \qquad \mathcal{D}(z) = (k+1)\delta + k(k+1)a^2.$$

In particular $\delta \geq 0$ makes these active directions strictly positive. If all vectors are orthogonal to z, then $\mathcal{D} = \delta$. An irreducible B of rank at least two has no zero-variance active direction; the all-parallel rank-one case is an exception.

For active unit h and kernel unit κ, write $z_p = \sqrt{p}\, h + e^{i\theta}\sqrt{1-p}\,\kappa$. Direct homogeneity of the two terms in (20) gives

$$\mathcal{D}(z_p) = (1-p^2)\delta + p^2\mathcal{D}(h) + 2a(h^*Th)p(1-p). \tag{22}$$

Thus $\delta \geq 0$ and the active-endpoint inequality suffice for every kernel weight and relative phase. Enlarging the Gram dimension represents positive Schur slack in the same way.

8 Relative stability and occupied-vertex domains

Matrix-norm derivative and perturbation bounds for permanents are developed in [13]. The following elementary comparison instead measures perturbations directly in unit Gram factors; we use its multiplicative form for the quantitative transfer below.

Lemma 8.1 (Relative Gram-permanent stability)**.** *Replace at most r of N unit vectors by unit vectors, each at distance at most ε. If P_0, P are their Gram permanents, then*

$$(1+\sqrt{N}\varepsilon)^{-2r} P_0 \leq P \leq (1+\sqrt{N}\varepsilon)^{2r} P_0.$$

Gram ranks and ambient dimension are unrestricted.

Proof. For one replacement let g be the symmetric product of the other $N-1$ vectors. Orthogonal projection gives $\|\operatorname{Sym}((v-w)\otimes g)\| \leq \|v-w\|\|g\|$. On each homogeneous degree define $c_w^* g = \sqrt{N}\operatorname{Sym}(w\otimes g)$, with c_w its graded adjoint. For unit w the classical identity is $\|c_w^* g\|^2 = \|g\|^2 + \|c_w g\|^2 \geq \|g\|^2$; see the bosonic conventions in [5, 6]. The permanent norm formula gives

$$|\sqrt{P(v)} - \sqrt{P(w)}| \leq \sqrt{N}\|v-w\|\sqrt{P(w)}.$$

Interchange v, w and apply the two ratio bounds successively. The products are nonzero, for example by the product-of-linear-forms representation. The vacuum case $N = 1$ is included. □

Theorem 8.2 (Transfer from an orthogonal complement)**.** *Let A be the Gram matrix of unit vectors $w_1, \dots, w_n, z$, and set $p_i = |w_i^* z|^2$. Suppose $0 \leq \tau < 1/2$, each p_i belongs to $[0,\tau] \cup [1-\tau, 1]$, and exactly $k \geq 1$ belong to the second interval. Replace those k vectors by their closest phase multiples of z, and the others by their normalized projections onto $z^\perp$. Let C be the Gram matrix of the latter $n-k$ vectors. If $\operatorname{per}|C|^2 \leq (\operatorname{per} C)^2$, then*

$$\frac{\operatorname{per}|A|^2}{(\operatorname{per} A)^2} \leq \frac{(1+\sqrt{2(n+1)\tau})^{6n}}{(k+1)!}. \tag{23}$$

The empty complement has permanent one.

Proof. Each ordinary replacement has distance at most $\sqrt{2\tau}$: its squared distance is $2 - 2\sqrt{p_i}$ or $2 - 2\sqrt{1-p_i}$. For doubled unit vectors $d(v) = v \otimes \bar{v}$,

$$\|d(v) - d(w)\|^2 = 2(1 - |v^* w|^2).$$

For these particular projections the doubled squared distances are $2(1-p_i)$ or $2p_i$, so the same bound holds. Put $\rho = \sqrt{2(n+1)\tau}$. With $P = \operatorname{per} A$ and $Q = \operatorname{per}|A|^2$, Lemma 8.1 gives $P \geq (1+\rho)^{-2n} P_0$ and $Q \leq (1+\rho)^{2n} Q_0$. The reference is $J_{k+1} \oplus C$, hence $Q_0/P_0^2 \leq 1/(k+1)!$. Combine the inequalities. □

Corollary 8.3 (A polynomial occupation radius)**.** *Under the complementary hypothesis in Theorem 8.2, if* $\tau \leq [1152n^2(n+1)]^{-1}$*, then* $\operatorname{per}|A|^2 \leq \frac{2}{3}(\operatorname{per} A)^2$.

Proof. Now $\rho \leq 1/(24n)$ and $(1+\rho)^{6n} \leq \sum_{j\geq 0}(6n\rho)^j \leq 4/3$. Use $(k+1)! \geq 2$. □

The complementary hypothesis is automatic when $n - k \leq 4$ by the complex small-order result [2]. For $n = 6, k = 1$ it instead requires the order-five statement, supplied by the self-contained reconstruction in Lemma 11.1 below of the coefficient argument in [3]. The general transfer retains its stated hypothesis. Exact rational comparisons in (23) give the ratio ceiling $7/8$ at these larger order-seven radii:

k	1	2	3	4	5	6
τ	1/57344	1/8064	1/2016	1/896	1/504	1/224

Choose respectively $\rho = 1/64, 1/24, 1/12, 1/8, 1/6, 1/4$. At the uniform radius $\tau_0 = 1/57344$,

$$(65/64)^{-24} - \tfrac{1}{2}(65/64)^{12} > 1/12, \qquad (65/64)^{36}/2 < 7/8.$$

The reference permanent $P_0 = (k+1)!\operatorname{per} C \geq 2$, by the classical PSD permanent diagonal bound [15]. Thus $\mathcal{D} \geq P_0^2/12 \geq 1/3$ on this uniform domain. These estimates are independent of positive Gram/cofactor eigenvalues. The all-low vertex is excluded. The conclusions have their stated finite sufficient domains.

For an active unit direction at order six, let $s = \lambda^+_{\min}(B)$ and $\tau = \min\{s/12, 1/57344\}$. If $E < \mu\tau(1-\tau)$, Proposition 7.1 forces every occupation outside $[\tau, 1-\tau]$. The active frame bound $\sum_i p_i \geq s$ excludes the all-low alternative. Thus the occupied vertex theorem applies, with its complementary hypothesis, and gives $\mathcal{D} \geq 1/3$. Only this conversion from variance uses s, μ; the direct geometric domain does not. In this small domain the surrogate is positive too: $E \leq \|L\| \sum_i p_i(1-p_i) \leq 4320/57344 < 1/3$, since $\operatorname{Tr} L \leq 6 \cdot 5! = 720$.

Theorem 8.4 (Neighborhoods of repeated orthogonal blocks)**.** *Let* $B_0 = \bigoplus_j J_{m_j}$ *have order n, with some* $m_j \geq 2$*. For any complex PSD correlation matrix A of order* $n+1$ *with leading block B, if*

$$\max_{i,j} |B_{ij} - (B_0)_{ij}| \leq \frac{1}{2304n^3(n+1)},$$

then $\operatorname{per}|A|^2 \leq \frac{2}{3}(\operatorname{per} A)^2$*. Gram ranks and the normalized border are unrestricted.*

Proof. At B_0 one has $a_0 = c_0 = \prod_j m_j! \geq 2$ and $U_0 = T_0$: $L_0 = R_0$ and $L_0 \circ B_0 = R_0$ by block cofactors. For every unit border vector, $p^{\mathsf{T}} L_0 p \leq z^* U_0 z = t$, so $Q_0 = a_0 + p^{\mathsf{T}} L_0 p \leq a_0 + t = P_0$. Thus $Q_0/P_0^2 \leq 1/a_0 \leq 1/2$. If the leading-block entrywise distance is at most d, then $\|B - B_0\| \leq nd$. Square-root factors differ rowwise by at most $\varepsilon = \sqrt{nd}$, by the PSD square-root bound below. Pad the factors to represent any admissible border by the same unit vector in both Gram systems. Their doubled-row distances are at most 2ε. Lemma 8.1 bounds the new ratio by

$$\frac{(1 + 2\sqrt{n+1}\varepsilon)^{6n}}{a_0}.$$

At the stated radius $2\sqrt{n+1}\varepsilon \leq 1/(24n)$; the same geometric-series estimate as Corollary 8.3 gives at most $4/(3a_0) \leq 2/3$. □

Corollary 8.5 (A neighborhood of a rank-five base)**.** *Let A be any complex PSD correlation matrix of order seven, with leading block B of order six. If*

$$\max_{i,j} |B_{ij} - (J_2 \oplus I_4)_{ij}| \le 1/884736,$$

then $\operatorname{per}|A|^2 \le \frac{7}{8}(\operatorname{per} A)^2$. *There is no rank assumption on A or B.*

Proof. At the reference base $B_0 = J_2 \oplus I_4$, one has $a = c = 2$ and $U = T$ by direct block cofactors. For every unit border vector in any ambient dimension, (21) gives

$$\tfrac{1}{2}(a+t)^2 - c - p^{\mathsf{T}} L p = t + \tfrac{1}{2}t^2 + E \ge 0.$$

Thus every reference border has ratio at most $1/2$. If its six unit base vectors are each perturbed by at most ε, keep the same unit border vector. The doubled-vector distances are at most 2ε. Lemma 8.1, with $N = 7, r = 6$, bounds the new ratio by

$$\tfrac{1}{2}(1 + 2\sqrt{7}\varepsilon)^{12}(1 + \sqrt{7}\varepsilon)^{24} \le \tfrac{1}{2}(1 + 6\varepsilon)^{36} < 7/8 \quad (\varepsilon \le 1/384).$$

For the stated entrywise radius, $\|B - B_0\| \le 6/884736$ and the PSD square-root inequality gives $\|\sqrt{B} - \sqrt{B_0}\| \le \sqrt{\|B - B_0\|} \le 1/384$. Its rowwise bound compares unit Gram factors. Pad them, if necessary, by an unused coordinate to represent arbitrary normalized border slack. Isometric equivalence of Gram realizations lets the same border vector be used in both factors. Hence the comparison covers every admissible A, including singular bases and full-rank borders. □

The square-root bound used here follows directly from operator monotonicity: if $\|B - B_0\| \le d$, then $\sqrt{B} \le \sqrt{B_0 + dI} \le \sqrt{B_0} + \sqrt{d}\, I$, and similarly in reverse. The neighborhood is a quantitative application.

9 Occupation-box certificates for the actual quartic

Return to $\mathcal{D}$ in (20). This section also holds for unnormalized B, with $0 \le p_i \le B_{ii}$. For real y set

$$D(y) = \delta I + 2aT - 2V^* \operatorname{diag}(Ly) V, \qquad F_y(z) = y^{\mathsf{T}} L y + z^* D(y) z + (z^* T z)^2.$$

Then $F_y(z) = \mathcal{D}(z) + (p - y)^{\mathsf{T}} L (p - y)$. Consequently

$$\min_{\|z\|=1} \mathcal{D}(z) = \min_{0 \le y_i \le B_{ii}} \left\{ y^{\mathsf{T}} L y + \max_{\xi \in [\lambda_{\min} T, \lambda_{\max} T]} \lambda_{\min}(D(y) + 2\xi T - \xi^2 I) \right\}. \tag{24}$$

The compact product minimum is exact because $y = p$ is allowed. At each fixed y, the inner scalar dual follows from convexity of the complex numerical range of $D(y) + iT$ and minimax for the convex function $d + t^2$; cf. [16]. The elementary scalar identity is $t^2 = \max_\xi (2\xi t - \xi^2)$. There is no interchange of the outer minimum and inner maximum in (24).

Proposition 9.1 (Finite box certificate)**.** *Cover $\prod_i [0, B_{ii}]$ by finitely many boxes of center h and halfwidth $r \ge 0$. A box is certified if some real shift ξ gives*

$$[\delta + h^{\mathsf{T}} L h - r^{\mathsf{T}} L r - \xi^2] I + 2(a + \xi) T - 2V^* \operatorname{diag}(Lh) V \succeq 0. \tag{25}$$

If every box is certified, then $\mathcal{D}(z) \ge 0$ for all unit complex z. For a strictly positive fixed-base instance, a sufficiently fine finite cover admits these certificates. With rational real and imaginary input coordinates, the cover and shifts can be rational.

Proof. All entries of L are nonnegative. If $|p - h| \le r$, then $(p - h)^{\mathsf{T}} L (p - h) \le r^{\mathsf{T}} L r$. Use this with $F_h - \mathcal{D} = (p - h)^{\mathsf{T}} L (p - h)$ and $(t - \xi)^2 \ge 0$ to obtain (25) as a lower bound. If $\min \mathcal{D} = m > 0$, every $F_h \ge m$. The exact fixed-center dual supplies a shift attaining that margin before the box error. A fine mesh with $r^{\mathsf{T}} L r < m/2$ leaves a strict PSD margin. Continuity permits rational nearby shifts. This is finite existence at one strictly positive base; no uniform complexity bound or termination at zero margin follows. □

Proposition 9.2 (The all-low box)**.** *For a correlation base, put* $J = V^* \operatorname{diag}(L\mathbf{1})V$*. Then* T *and* J *have kernel exactly* $\ker V$*. Every fixed base therefore admits* $\eta > 0$ *with* $2aT - \eta J \geq 0$*. If also* $\delta \geq 0$*, this test certifies the entire box* $0 \leq p_i \leq \eta$*, with* $\mathcal{D} \geq \delta + t^2$*.*

Proof. The omitted-product argument for the original unit Gram vectors gives the same line-class range description for R, after phases make parallel rows equal. The image of V lies in that range, so R is positive on it and $\ker T = \ker V$. Under a row gauge $B = DB_0D^*$, $R(B) = DR(B_0)D^*$ and $V = DV_0$, preserving this argument. Every diagonal entry of L is at least one by the PSD permanent diagonal bound; hence J has the same kernel. Positivity on the active subspace gives an admissible η. Finally, entrywise nonnegativity of L gives $p^{\mathsf{T}}Lp \leq \eta p^{\mathsf{T}}L\mathbf{1} = \eta z^*Jz$. Substitute into (20). □

There is a conditional finite-completeness extension through kernel zeros. Suppose $\delta \geq 0$ and $\mathcal{D}(z) > 0$ whenever $Vz \neq 0$. Use Proposition 9.2 on an all-low box. On the remaining nonnegative auxiliary centers with $\max_i h_i \geq \eta/2$, each F_h is strictly positive on the whole unit sphere: off the kernel this follows from $F_h \geq \mathcal{D} > 0$, and on it $F_h = \delta + h^{\mathsf{T}}Lh > 0$ because $L_{ii} \geq 1$. Compactness over those centers gives a common positive F_h margin. Fine subdivision and the fixed-center dual then give finitely many spectral leaves. This assumes the active-endpoint sign.

An adaptive cover may also accept a box by Theorem 8.2 if every coordinate is wholly low or high at a valid radius and at least one is high. Its complementary hypothesis must still be supplied. The all-low box cannot be accepted by that theorem. Standard rigorous partition methods [17] motivate this organization; the quadratic conjugacy and scalar numerical-range dual are classical. The reusable point here is a certificate of the actual quartic, without first proving the stronger cofactor surrogate. The supplied adaptive examples illustrate subdivision and exact checking. In particular, the first rational full-support, phase-bearing rank-six base in `adaptive_certificates.json` has a two-leaf proof of $\mathcal{D} \geq 13/20$. An exact Rayleigh witness rules out that margin for every single-full-box shift, while the exact shifted cofactor test also certifies $13/20$ on the same base. This is a subdivision example, rather than evidence that retained variance is necessary there.

10 Uniform cofactor-weighted variance

The coefficient $\lambda^+_{\min}(L)$ in Proposition 7.1 can degenerate when Gram lines merge. The following trace-contraction estimate removes that coefficient. We first state it for trace-one operators; its variance must then be the operator variance, rather than a Bernoulli expression.

Theorem 10.1 (Trace-one operator variance)**.** *Let* $P_1, \dots, P_n$ *be positive semidefinite trace-one operators,* $n \geq 2$*, and let* H_0 *be the span of their ranges, of dimension* r*. For any unit* z*, set*

$$Q_z = zz^*, \quad \pi = \|P_{H_0}z\|^2, \quad h_{ij} = \operatorname{Tr}(P_iP_j), \quad L_{ij} = \operatorname{per} h(j \mid i), \quad \rho_i = (I - Q_z)P_iz,$$

and put

$$v = \sum_i \|\rho_i\|^2 = \sum_i \left[z^*P_i^2z - (z^*P_iz)^2\right], \qquad E = \sum_{i,j} L_{ij}\langle\rho_i, \rho_j\rangle.$$

Writing $y = \sqrt{E/(n-1)!}$ *and* $k = (n-1)r^{(n-2)/2}\sqrt{\pi(r-\pi)}$*, one has*

$$v \leq r^{n-1}y^2 + ky. \tag{26}$$

In particular, for $v > 0$*,*

$$E \geq (n-1)! \left[\frac{2v}{k + \sqrt{k^2 + 4r^{n-1}v}}\right]^2. \tag{27}$$

For $v = 0$ *the conclusion is* $E \geq 0$*. The support dimension* r *need not be bounded by* n*.*

Proof. Work in the Hilbert–Schmidt space $K = \mathrm{End}(H_0)$. Let q_i be the averaged symmetric tensor product of the $n-1$ operators other than P_i. The permanent norm identity gives $\langle q_i, q_j\rangle = L_{ij}/(n-1)!$; here h and L are real symmetric. Hence, for $Y = \sum_i \rho_i \otimes q_i$,

$$E = (n-1)!\|Y\|^2.$$

The trace functional ℓ on K has norm $\sqrt{r}$ and $\ell(P_i) = 1$. Full contraction and one uncontracted slot give

$$S_1 = \sum_i \rho_i = (I \otimes \ell^{\otimes(n-1)})Y,$$
$$Z = \sum_i \rho_i \otimes \sum_{j\neq i} P_j = (n-1)(I \otimes I_K \otimes \ell^{\otimes(n-2)})Y.$$

The factor $n-1$ compensates for averaged symmetrization. Therefore

$$\|S_1\| \le r^{(n-1)/2}\|Y\|, \qquad \|Z\| \le (n-1)r^{(n-2)/2}\|Y\|.$$

On $z^\perp \otimes K$ the complex-linear functional $F(s \otimes P) = z^* P s$ has squared norm

$$\|F\|^2 = \|P_{H_0} z\|^2 \,\mathrm{Tr}\big[P_{H_0}(I - zz^*)\big] = \pi(r-\pi).$$

For example, this follows by summing its squared coefficients over matrix units of K and an orthonormal basis of $z^\perp$ in $\mathrm{span}(H_0, z)$. Extra ambient directions contribute zero. Hermiticity, with $P_\Sigma = \sum_i P_i$, gives

$$F(S_1 \otimes P_\Sigma) = \|S_1\|^2, \qquad F\left(\sum_i \rho_i \otimes P_i\right) = v.$$

Since $S_1 \otimes P_\Sigma - Z = \sum_i \rho_i \otimes P_i$, the exact cancellation identity is $v = \|S_1\|^2 - F(Z)$. The norm bounds prove (26); its positive quadratic root gives (27). The proof applies to the stated mixed-state class. □

Corollary 10.2 (A first six-factor specialization)**.** *For the variance in* (21)*, let* $v = \sum_i p_i(1-p_i)$*. For every complex correlation base of order six and every unit border vector in any ambient dimension,*

$$E \ge \frac{v^2}{1600}.$$

If $\mathrm{rank}\, B \le 5$*, then* $E \ge v^2/600$*. Both statements include every kernel mixture and all rank limits. The same constants hold for six trace-one operators with joint support dimension at most six or five, respectively, with their actual operator variance* v*.*

Proof. Trace-one positivity implies $0 \le P_i \le I$ and $\mathrm{Var}_z(P_i) \le 1/4$, so $v \le 3/2$. For $r \ge 2$, $\pi(r-\pi) \le r-1$. Put $k_r = 5r^2\sqrt{r-1}$. Theorem 10.1 gives

$$E \ge \frac{480v^2}{[k_r + \sqrt{k_r^2 + 6r^5}]^2}.$$

At $r = 6$, $k_r^2 = 162000$ and $k_r^2 + 6r^5 = 208656$; their square roots are less than 403 and 457. Since $860^2 < 480 \cdot 1600$, the first bound follows. For $2 \le r \le 5$ the denominator increases with r; at $r = 5$ it is less than $250 + 286 = 536$, and $536^2 < 480 \cdot 600$. For $r = 1$ every operator is the same rank-one projector, and $E = n!v = 720v$, which covers both constants. □

The exponent two cannot be decreased in a uniform positive power-law bound. Indeed, for $s > 0$ take the first two unit Gram vectors to be $(e_0 \pm s e_1)/\sqrt{1+s^2}$, the other four mutually orthogonal and orthogonal to e_0, e_1, and take $z = e_1$. Directly,

$$v = \frac{2s^2}{(1+s^2)^2}, \qquad E = \frac{8s^4}{(1+s^2)^4} = 2v^2.$$

The bases have rank six for $s > 0$ and rank five at $s = 0$. This proves optimality of the exponent, not of our numerical constants.

11 A quantitative base-defect reserve

We include the small-order input explicitly. The coefficient induction below is the method of Li–Zhao–Luo [3]; its inclusion identifies the dependency of the quantitative extraction that follows.

Lemma 11.1 (Small-order coefficient induction)**.** *For every complex positive semidefinite matrix A of order at most six,* $\operatorname{per}|A|^2 \le (\operatorname{per} A)^2$.

Proof. For PSD matrices M_j, every coefficient of $\operatorname{per}(\sum_j t_j M_j)$ is nonnegative: realize their Gram vectors in orthogonal labelled spaces, expand the symmetric product, and group by label occupations. The coefficients are factorial multiples of squared norms of the orthogonal occupation components.

A zero diagonal of $A \succeq 0$ gives a zero row and trivial permanents. Otherwise diagonal congruence reduces to diagonal one; both sides scale by the same factor. Write $A = \left(\begin{smallmatrix} B & x \\ x^* & 1 \end{smallmatrix}\right)$, $H = B - xx^* \succeq 0$, $X = xx^*$, and, if B has order m, set

$$F(s) = \operatorname{per}(H + sX) = \sum_{k=0}^{m} \alpha_k s^k,$$
$$\Gamma(s,t) = \operatorname{per}\big(H \circ \bar{H} + s(H \circ \bar{X} + \bar{H} \circ X) + tX \circ \bar{X}\big) = \sum_{j,k} \gamma_{jk} s^j t^k.$$

Every coefficient is nonnegative by the preceding argument and the Schur product theorem. Border expansion and rank-one differentiation give

$$\operatorname{per} A = \sum_k (k+1)\alpha_k, \qquad \operatorname{per}|A|^2 = \sum_{j,k} (k+1)\gamma_{jk}, \qquad \Gamma(s, s^2) = \operatorname{per}|H + sX|^2.$$

If $m \le 5$, then $(\sqrt{2})^k \le k+1$ for $0 \le k \le m$, whereas $k+1 \le 2^k \le 2^{j/2+k}$ for all $j, k \ge 0$. Assuming the assertion in order m, these comparisons yield

$$\operatorname{per}|A|^2 \le \Gamma(\sqrt{2}, 2) \le F(\sqrt{2})^2 \le (\operatorname{per} A)^2.$$

Start from scalar equality and induct to order six. At $k = 6$ the last ordinary coefficient comparison fails ($8 > 7$); this argument gives no unrestricted order-seven theorem. □

Lemma 11.2 (Sharp nonzero occupation mass)**.** *Let $f = \operatorname{Sym}(v_1 \otimes \cdots \otimes v_m)$ for unit vectors v_i, and let e be unit. If $q = \max_i |v_i^* e|^2$ and P_0 projects onto $\operatorname{Sym}^m(e^\perp)$, then*

$$\|(I - P_0)f\|^2 \ge q^m \|f\|^2.$$

The constant q^m is sharp among such products.

Proof. Choose a factor v attaining q, let $M = v \operatorname{Sym}^{m-1} H$, and let $N = \operatorname{Sym}^m(e^\perp)$. On $e^\perp$, projection onto $v^\perp$ has least singular value at least $\sqrt{q}$. Its symmetric tensor power is projection onto $M^\perp$ restricted to N, with least singular value at least $q^{m/2}$. Thus $P_N P_M P_N \le (1 - q^m) P_N$. Since $f \in M$, $\|P_N f\|^2 \le (1 - q^m)\|f\|^2$.

For sharpness take unit $u \perp e$, $0 < q < 1$, and $v = \sqrt{1-q}\,u + \sqrt{q}\,e$. In the binary polynomial model, $P_M u^m = u^m - [qu - \sqrt{q(1-q)}e]^m$ factors as

$$\prod_{\omega^m = 1} [(1 - \omega q)u + \omega\sqrt{q(1-q)}e].$$

The squared e-overlap of each normalized factor is $q(1-q)/(1 + q - 2q \operatorname{Re}\omega) \le q$, with equality at $\omega = 1$. Moreover $\|P_M u^m\|^2 = 1 - q^m$ and its zero-mode projection is $(1 - q^m)u^m$, giving nonzero-mode fraction exactly q^m. The endpoints follow directly. □

Theorem 11.3 (Coherence reserve)**.** *For a complex correlation matrix B of order six, put $\eta = \max_{i\neq j} |B_{ij}|^2$, $r_0 = \eta^5$, and*

$$d_0 = \frac{(6-4\sqrt{2})r_0}{1+5r_0}, \qquad \bar{d}_0 = \frac{r_0}{3(1+5r_0)}.$$

Then

$$\frac{\delta}{a^2} \geq 2d_0 - d_0^2 \geq \overline{D}(\eta) := 2\bar{d}_0 - \bar{d}_0^2.$$

In particular $\delta = 0$ if and only if $B = I_6$.

Proof. Reorder a pair attaining η so that one is the last vector, and split the other five factors into its mode and perpendicular part. Use $F(s) = \operatorname{per}(H + sxx^*) = \sum_{k=0}^{5} \alpha_k s^k$ as in Lemma 11.1. Put $\rho = \sum_{k\geq 1} \alpha_k / F(1)$. The coefficients are the ordinary occupation masses; Lemma 11.2 gives $\rho \geq \eta^5$. With $b = F(\sqrt{2})$, the induction proof gives $c \leq b^2$, while

$$a = \sum_k (k+1)\alpha_k, \quad a - b \geq (6-4\sqrt{2})\rho F(1), \quad a \leq (1+5\rho)F(1).$$

Here $6-4\sqrt{2}$ is the minimum of $k+1-2^{k/2}$ for $1 \leq k \leq 5$. Since $\rho/(1+5\rho)$ increases, $(a-b)/a \geq d_0$ and $c/a^2 \leq (1-d_0)^2$. Finally $6-4\sqrt{2} > 1/3$ follows from $289 > 288$. At $\eta = 0$, $B = I_6$ directly. □

Corollary 11.4 (Uniform low occupations and kernel weights)**.** *Let $p_{\max} = \max_i |(Vz)_i|^2$. Then*

$$\frac{\mathcal{D}(z)}{a^2} \geq \overline{D}(\eta) - 36[1 - \overline{D}(\eta)]p_{\max}^2. \tag{28}$$

If $\eta \geq 1/25$ (in particular if rank $B \leq 5$*), then*

$$\delta \geq \frac{a^2}{15000000}, \qquad p_{\max} \leq \frac{1}{30000} \Longrightarrow \mathcal{D}(z) \geq \frac{a^2}{37500000}.$$

For every active unit h and kernel unit κ, all vectors $z_p = \sqrt{p}\, h + e^{i\theta}\sqrt{1-p}\,\kappa$ with $0 \leq p \leq 1/10000$ satisfy $\mathcal{D}(z_p) \geq a^2/150000000$ under the same coherence condition.

Proof. The unit-vector creator lower bound gives $L_{ii} \leq c$; positivity of L gives $0 \leq L_{ij} \leq c$. Thus $p^{\mathsf{T}} L p \leq 36 c p_{\max}^2$; drop the nonnegative $2at + t^2$ in (20) to obtain (28). If rank $B \leq 5$, $\operatorname{Tr} B^2 \geq 36/5$ and $\operatorname{Tr} B^2 \leq 6 + 30\eta$ give $\eta \geq 1/25$. At that endpoint $\bar{d}_0 = 1/29296890$ and $2\bar{d}_0 - \bar{d}_0^2 > 1/15000000$. The low-occupation margin follows from $1/15000000 - 36/30000^2 = 1/37500000$. Finally $p(h)^{\mathsf{T}} L p(h) \leq h^* U h \leq \operatorname{Tr} U = 6c$, so homogeneity gives $\mathcal{D}(z_p)/a^2 \geq 1/15000000 - 6p^2 \geq 1/150000000$. This includes every relative phase and both endpoints of the stated band. □

12 Uniform matrix and occupation-box certificates

Proposition 12.1 (Low-coherence matrix compensation)**.** *For a complex correlation matrix B of order six with $|B_{ij}| \leq 1/15$ for $i \neq j$,*

$$2aR - L \circ B \geq \tfrac{1}{12} I_6, \qquad 2aT - U \geq \tfrac{1}{12} V^* V.$$

Consequently every admissible border satisfies $\mathcal{D}(z) \geq \delta + \frac{1}{12}\sum_i p_i + t^2 + E$.

Proof. Put $\rho = 1/15$ and

$$f(x) = x + 4x^2 + 18x^3 + 44x^4 + 53x^5, \quad d(x) = 1 + 10x^2 + 20x^3 + 45x^4 + 44x^5.$$

Counting permutations by their number of off-diagonal factors gives $|R_{ij}| \leq f(\rho)$ for $i \neq j$, $L_{ii} \leq d(\rho^2)$, and $L_{ij} \leq f(\rho^2)$ for $i \neq j$. The corresponding counts are $(1, 4, 18, 44, 53)$ for an off-diagonal five-row

cofactor and $(1, 0, 10, 20, 45, 44)$ for a diagonal one. The creator lower bound gives $R_{ii} \geq 1$ and $a \geq 1$. Hermitian row-sum bounds therefore yield

$$R \geq r_* I, \quad L \circ B \leq s_* I, \quad r_* = 1 - 5f(\rho), \quad s_* = d(\rho^2) + 5\rho f(\rho^2).$$

Direct rational evaluation gives

$$r_* = \frac{82987}{151875} > 0, \qquad 2r_* - s_* = \frac{31528452158}{345990234375} > \frac{1}{12}.$$

Thus $2aR - L \circ B \geq (2r_* - s_*)I$; congruence by V and (21) prove the assertions. Lemma 11.1 supplies $\delta \geq 0$. No inverse or singular-border assumption is used. □

This base domain has $B \geq (2/3)I$ by Gershgorin. It contains no rank-five base: the preceding trace bound forces an off-diagonal modulus at least $1/5$ at rank at most five. Outside the certified domain, Theorem 11.3 gives a uniform positive reserve with $\bar{d}_0 = 1/[3(225^5 + 5)]$.

Proposition 12.2 (Variance-corrected boxes)**.** *For a six-row correlation base suppose $\ell_i \leq p_i \leq h_i$, with $0 \leq \ell_i \leq h_i \leq 1$. Put*

$$v_{\min} = \sum_i \min\{\ell_i(1 - \ell_i), h_i(1 - h_i)\}, \qquad C = \begin{cases} 600, & \operatorname{rank} B \leq 5, \\ 1600, & \textit{otherwise}. \end{cases}$$

For any real τ, the matrix test

$$\left(\delta - \tau^2 - m + \frac{v_{\min}^2}{C}\right) I + 2(a + \tau)T - U \geq 0$$

certifies $\mathcal{D}(z) \geq m$ on that feasible occupation box.

Proof. Concavity of $s(1 - s)$ gives $v \geq v_{\min}$. The scalar square $(t - \tau)^2 \geq 0$ lower-bounds $Q(z)$ by the quadratic form of $(\delta - \tau^2)I + 2(a + \tau)T - U$. Add Corollary 10.2 and use $\mathcal{D} = Q + E$. □

The test improves a shifted cofactor margin without a positive cofactor eigenvalue assumption. The matrix test is a sufficient criterion.

13 Orthogonal trace sectors and frame variance

The trace contractions in Theorem 10.1 come from orthogonal tensor sectors. Accounting for their joint norm improves the constants and retains the variance of the sum of the operators.

Theorem 13.1 (Joint trace-sector bound)**.** *Use the trace-one operator setting of Theorem 10.1, and put $m = n - 1$, $\mathsf{F} = \sum_i P_i$,*

$$s = z^* \mathsf{F} z, \qquad \mathcal{A} = z^* \mathsf{F}^2 z - s^2, \qquad b = \frac{m}{r}(1 - \pi)s, \qquad f_0^2 = \frac{\pi(r - 1)(r + 1 - \pi)}{r}.$$

If $f_0 > 0$, then

$$\frac{E}{m!} \geq \frac{\mathcal{A}}{r^m} + \frac{(\mathcal{A} - b - v)^2}{m r^{m-1} f_0^2}. \tag{29}$$

If $f_0 = 0$, the second quotient is omitted and $\mathcal{A} - b - v = 0$. In particular, for $r \geq 2$,

$$\frac{E}{m!} \geq \frac{\mathcal{A}}{r^m} + \frac{(v - \mathcal{A})_+^2}{m r^{m-1}(r - 1)}. \tag{30}$$

These statements retain arbitrary kernel components of z and do not require rank-one P_i.

Proof. In the Hilbert–Schmidt space write $e = I_{H_0}/\sqrt{r}$ and $P_i = e/\sqrt{r} + D_i$, with D_i traceless. Let Y and S_1 be the omitted-product tensor and its full contraction from Theorem 10.1, and set

$$Z_0 = \sum_i \rho_i \otimes \sum_{j \neq i} D_j.$$

Projecting the last m slots of Y onto zero or one traceless slot gives orthogonal components of squared norms

$$\|Y_0\|^2 = \frac{\|S_1\|^2}{r^m}, \qquad \|Y_1\|^2 = \frac{\|Z_0\|^2}{mr^{m-1}}.$$

The factor $1/m$ follows because the m possible traceless slots are orthogonal under averaged symmetrization. Also $\|S_1\|^2 = \mathcal{A}$. Restricting the functional $F(u \otimes P) = z^* P u$ to traceless matrices removes squared norm $\pi(1-\pi)/r$ from its full squared norm $\pi(r-\pi)$. Its remaining squared norm is exactly f_0^2.

The identity-slot contraction must retain the support projection: $F(S_1 \otimes I_{H_0}) = (1-\pi)s$. Thus the exact cancellation is

$$v = \mathcal{A} - b - F(Z_0).$$

Cauchy–Schwarz and $E = m!\|Y\|^2$ prove (29), including its zero-norm convention. For $r \geq 2$, $f_0^2 \leq r - 1$ and $b \geq 0$ give $v \leq \mathcal{A} + \sqrt{r-1}\|Z_0\|$, proving (30). □

Corollary 13.2 (Six-factor frame reserve)**.** *For six trace-one operators of joint support dimension at most six,*

$$E \geq \frac{v^2}{270} + \frac{7}{1620}\mathcal{A}. \tag{31}$$

At support dimension at most five,

$$E \geq \frac{6}{625}(v^2 + \mathcal{A}). \tag{32}$$

For unit Gram vectors, $v = \sum_i p_i(1-p_i)$ and $\mathsf{F} = V^*V$. These inequalities improve the round constants in Corollary 10.2.

Proof. Embed the support, if necessary, into a containing space of dimension $R = 6$ or 5. The preceding proof needs only containment, not that the ranges span the containing space. Since $v \leq 3/2$, (30) gives

$$E \geq \alpha\mathcal{A} + \beta(v - \mathcal{A})_+^2, \qquad \alpha = \frac{120}{R^5}, \quad \beta = \frac{24}{R^4(R-1)}.$$

For $\mathcal{A} \leq v$, expansion yields $E \geq \beta v^2 + (\alpha - 3\beta)\mathcal{A}$ because $2v \leq 3$. For $\mathcal{A} \geq v$, the same conclusion follows from $3\mathcal{A} \geq v^2$. Substitution of $R = 6, 5$ gives the displayed constants. The argument includes rank one; directly in that case $E = 720v$. □

Proposition 13.3 (Frame-corrected occupation boxes)**.** *For a six-row correlation base let $0 \leq \ell_i \leq p_i \leq h_i \leq 1$ and $v_{\min} = \sum_i \min\{\ell_i(1-\ell_i), h_i(1-h_i)\}$. Suppose real $\lambda < \mu$ satisfy*

$$(\mathsf{F} - \lambda I)(\mathsf{F} - \mu I) \geq 0, \qquad \lambda \leq s_- := \sum_i \ell_i \leq s_+ := \sum_i h_i \leq \mu.$$

Set

$$A_{\min} = \min\{(s_- - \lambda)(\mu - s_-), (s_+ - \lambda)(\mu - s_+)\}.$$

Take $(\beta, \gamma) = (1/270, 7/1620)$, or $(6/625, 6/625)$ when rank $B \leq 5$*. For any real τ,*

$$[\delta - \tau^2 - M + \beta v_{\min}^2 + \gamma A_{\min}]I + 2(a+\tau)T - U \geq 0 \tag{33}$$

certifies $\mathcal{D}(z) \geq M$ throughout the feasible box.

Proof. The spectral condition implies $\mathcal{A} \geq (\lambda+\mu)s - \lambda\mu - s^2 = (s-\lambda)(\mu-s)$. Concavity gives $\mathcal{A} \geq A_{\min}$ on the box, while $v \geq v_{\min}$. Add Corollary 13.2 to the shifted lower bound for Q from $(t-\tau)^2 \geq 0$. □

If no spectral gap and box interval give a positive frame floor, one may use $A_{\min} = 0$. The matrix condition includes kernel eigenvalues; no inverse is used. For rational input and endpoints all checks in (33) are rational. This remains a sufficient local certificate, not a uniform cover of all bases.

14 Rank-one secants couple the base reserve

For a six-row unit Gram factor and unit z, set $x = Vz$, $X = xx^*$, $H = B - X \succeq 0$, and

$$F(\sigma) = \operatorname{per}(H + \sigma X), \qquad C(\sigma) = \operatorname{per}|H + \sigma X|^2.$$

The coefficients of both polynomials are nonnegative by the PSD occupation expansion in Lemma 11.1. The derivative identities are $F(1) = a$, $F'(1) = t = z^*Tz$, $C(1) = c$ and $C'(1) = 2u$, where $u = z^*Uz$.

Proposition 14.1 (Same-data secants)**.** *For every $\sigma > 1$ put $\beta_\sigma = (\sigma^6 - 1)/6$. Then*

$$u \le \frac{(a + \beta_\sigma t)^2 - c}{2(\sigma - 1)}. \tag{34}$$

Consequently, writing $k = t/a$ and $d = \delta/a^2$,

$$k \ge 6/7 \implies Q(z) \ge a^2/1000, \tag{35}$$

$$k \le d/24 \implies Q(z) \ge \delta/20. \tag{36}$$

Each conclusion gives the same lower bound for $\mathcal{D}$ plus E.

Proof. For $1 \le j \le 6$, the average $(1 + \sigma + \cdots + \sigma^{j-1})/j$ increases with j. Nonnegative coefficients of F therefore give $F(\sigma) \le a + \beta_\sigma t$. Convexity gives $C(\sigma) \ge c + 2(\sigma - 1)u$, while Lemma 11.1 gives $C(\sigma) \le F(\sigma)^2$. This proves (34).

At $\sigma = 5/4$, put $A_0 = 253/2048$, $B_0 = 18785783/33554432$. Expansion yields $Q/a^2 \ge -d + A_0 k + B_0 k^2$. The independent trace estimate $U \preceq 6cI$ gives $Q/a^2 \ge 7d + 2k + k^2 - 6$. Seven copies of the first lower bound and one of the second cancel d:

$$\frac{Q}{a^2} \ge \frac{(2 + 7A_0)k + (1 + 7B_0)k^2 - 6}{8}.$$

This increasing polynomial has value $28555209/3288334336 > 1/1000$ at $k = 6/7$. At $\sigma = 2$, (34) instead gives $Q/a^2 \ge d/2 - 17k/2 - 433k^2/8$. Since $0 \le d \le 1$, $k \le d/24$ makes this at least $(239/4608)d \ge d/20$. □

The low-occupation condition retains the actual base reserve. A proved floor $d \ge D$ permits the uniform sufficient condition $k \le D/24$. For example, Corollary 11.4 gives $k \le 1/360000000$ at rank at most five. These small constants are conservative; the high-occupation domain has no coherence or spectral-gap condition. Neither statement covers all intermediate occupations.

Corollary 14.2 (Retaining the top ordinary occupation)**.** *Put $\theta = 720 \prod_i p_i/a$. If $k \ge 3/4$ and $\theta \le 1/50$, then $Q(z) \ge a^2/30$ and $\mathcal{D}(z) \ge a^2/30 + E$.*

Proof. If $F(\sigma) = \sum_{j=0}^{6} \alpha_j \sigma^j$, then $\alpha_6 = \operatorname{per} X = 720 \prod_i p_i$ exactly. Bounding degrees one through five by their endpoint chord gives

$$F(\sigma) \le a + b_\sigma t + e_\sigma \alpha_6, \quad b_\sigma = \frac{\sigma^5 - 1}{5}, \quad e_\sigma = \sigma^6 - 1 - 6b_\sigma > 0.$$

At $\sigma = 5/4$, $b_* = 2101/5120$ and $e_* = 7221/20480$. Write $A_* = 2 - 4b_*$, $B_* = 1 - 2b_*^2$. The same secant and trace combination gives

$$\frac{Q}{a^2} \ge J(k, \theta) := \frac{(2 + 7A_*)k + (1 + 7B_*)k^2 - 6 - 7(4e_*\theta + 4b_*e_*k\theta + 2e_*^2\theta^2)}{8}.$$

For $0 \le \theta \le 1/50$ this increases with $k \ge 0$: its derivative numerator is at least $2 + 7A_* - 28b_*e_*/50 = 5805352753/1310720000 > 0$. It decreases with $\theta \ge 0$. Thus its minimum in the stated region is $J(3/4, 1/50) = 156051912513/4194304000000 > 1/30$. □

All three domains permit arbitrary complex phases and exact rank-five boundaries. Along a kernel mixture of active weight p, k scales by p and θ by p^6; the conditions are required on the actual mixture, rather than automatically on its whole ray.

15 Retaining inherited small-order defect

The coefficient proof through order six loses a useful positive term if its smaller-order inequality is used only as a sign. For a correlation matrix A of order $m+1 \le 6$, use the notation H, X, F, Γ from Lemma 11.1 and put $M = H + \sqrt{2}X$, $P = \operatorname{per} A$, $b = \operatorname{per} M$. Write $\Delta_j(D) = (\operatorname{per} D)^2 - \operatorname{per}|D|^2$ for a matrix of order j. Then exactly

$$\Delta_{m+1}(A) = (P^2 - b^2) + \Delta_m(M) + [\operatorname{per}|M|^2 - \operatorname{per}|A|^2]. \tag{37}$$

All terms are nonnegative. If α_k and γ_{jk} are the ordinary coefficients of F and Γ, the two outer gaps are controlled without changing their common Gram origin by

$$P - b = \sum_k (k + 1 - 2^{k/2})\alpha_k \ge 0, \quad \operatorname{per}|M|^2 - \operatorname{per}|A|^2 = \sum_{j,k} (2^{j/2+k} - k - 1)\gamma_{jk} \ge 0.$$

Proposition 15.1 (Coherent inherited reserve)**.** *Let A be a correlation matrix of order six, choose any last-row split, and form the five-row matrix $M = H + \sqrt{2}xx^*$. Define its normalized coherence by*

$$\xi = \max_{i \ne j} \frac{|M_{ij}|^2}{M_{ii}M_{jj}}, \quad d_5 = \frac{(2-\sqrt{2})\xi^4}{1 + 4\xi^4}, \quad D_5 = 2d_5 - d_5^2.$$

Then

$$\frac{\Delta_6(A)}{(\operatorname{per} A)^2} \ge 1 - \left(\frac{\operatorname{per} M}{\operatorname{per} A}\right)^2 (1 - D_5) \ge D_5. \tag{38}$$

In particular, $\xi \ge 3/4$ implies

$$\Delta_6(A)/(\operatorname{per} A)^2 \ge 181359/1345600 > 1/9.$$

Proof. The diagonal of M is $1 + (\sqrt{2} - 1)|x_i|^2 > 0$. Apply diagonal normalization to M. The proof of Theorem 11.3, now with four omitted factors, uses the mass $\rho \ge \xi^4$ and the minimum gap $\min_{1 \le j \le 4}(j + 1 - 2^{j/2}) = 2 - \sqrt{2}$. It gives $\Delta_5(M)/(\operatorname{per} M)^2 \ge D_5$; the ratio is invariant under diagonal congruence. Insert this in (37) and use $b \le P$. Since $2 - \sqrt{2} > 1/2$, one may replace d_5 by $\bar{d}_5 = \xi^4/[2(1 + 4\xi^4)]$. At $\xi = 3/4$, $\bar{d}_5 = 81/1160$, giving the stated rational value. □

This floor is larger than the maximum $1/9$ of the single-coherence six-row floor in Theorem 11.3. It also explains a limitation of using occupation mass alone. For the sharp five-factor binary products in Lemma 11.2, with maximum mode overlap $q \le 1/24$, append that mode as the sixth factor. In M, the mode amplitudes are multiplied by $2^{1/4}$; after normalization their squared overlaps are at most $\sqrt{2}q$. All perpendicular components lie on one line, so every pair has inner-product modulus at least $1 - 2\sqrt{2}q > 7/8$. Thus $\xi > 3/4$, even though their nonzero-mode mass is only $q^5 \to 0$. The inherited defect therefore does not degenerate with that mass. This example explains the retained term within the known low-rank setting.

The inherited floor may be combined with the secant and frame bounds on the same actual data. Care is needed: the bound $Q/a^2 \ge -d + A_0 k + B_0 k^2$ decreases with d, so a lower floor for d cannot simply be substituted into it. With only an interval for d, minimize the maximum of this affine bound and $7d + 2k + k^2 - 6$ over that interval, and then add any proved frame/variance correction.

16 Forward reserves retain their common permanent

Continue with a six-row correlation base $B = VV^*$ and an arbitrary unit z, including a component in $\ker V$. Put $x = Vz$, $X = xx^*$, $H = B - X \ge 0$, $u = z^*Uz$, $\ell = p^{\mathsf{T}}Lp$, and, for $s > 1$,

$$M_s = H + sX, \quad F(s) = \operatorname{per} M_s, \quad C(s) = \operatorname{per}|M_s|^2, \quad \Delta_s = F(s)^2 - C(s).$$

Thus $E = u - \ell$ and $\mathcal{D} = \delta + 2at + t^2 - \ell$. Lemma 11.1 gives $\Delta_s \ge 0$. The ordinary and paired PSD occupation expansions give $F(1) = a$, $F'(1) = t$, $C(1) = c$ and $C'(1) = 2u$. All statements in this section concern these same actual matrices.

Normalize the strictly positive diagonal of M_s to obtain a correlation matrix K_s. For any selected index j, let y be its column with entry j omitted, and form

$$N_j = K_s(j \mid j) + (\sqrt{2}-1)yy^*, \qquad \xi = \max_{i\neq k} \frac{|(N_j)_{ik}|^2}{(N_j)_{ii}(N_j)_{kk}}.$$

Here N_j is precisely the five-row inherited child in Proposition 15.1. Its diagonal is positive even when the base is singular. Define the rational lower floor

$$d(\xi) = \frac{\xi^4}{2(1+4\xi^4)}, \qquad D(\xi) = 2d(\xi) - d(\xi)^2.$$

Diagonal-congruence invariance and that proposition imply $\Delta_s \geq D(\xi)F(s)^2$. No inverse or positive lower bound for a Gram eigenvalue is required.

Proposition 16.1 (Multiplicatively retained secant)**.** *Suppose* $0 \leq D \leq 1$ *is any proved floor for* $\Delta_s/F(s)^2$. *With* $\beta_s = (s^6-1)/6$, *one has*

$$u \leq \frac{(1-D)F(s)^2 - c}{2(s-1)} \leq \frac{(1-D)(a+\beta_s t)^2 - c}{2(s-1)}. \tag{39}$$

At $s = 5/4$, *set*

$$\beta = \frac{3843}{8192}, \quad A_0 = \frac{253}{2048}, \quad B_0 = \frac{18785783}{33554432}, \qquad k = \frac{t}{a},$$
$$f(k) = \frac{(2+7A_0)k + (1+7B_0)k^2 - 6}{8}.$$

Then the same-data lower bound is

$$\frac{Q(z)}{a^2} \geq L_D(k) := f(k) + \frac{7}{4}D(1+\beta k)^2, \qquad \frac{\mathcal{D}(z)}{a^2} \geq L_D(k) + \frac{E}{a^2}. \tag{40}$$

Proof. Convexity gives $c + 2(s-1)u \leq C(s) = (1-\Delta_s/F(s)^2)F(s)^2$. Insert the defect floor before using the ordinary chord $F(s) \leq a + \beta_s t$ from Proposition 14.1. The substitution is valid because $1 - D \geq 0$. At $s = 5/4$, expansion yields

$$Q/a^2 \geq -\delta/a^2 + A_0 k + B_0 k^2 + 2D(1+\beta k)^2.$$

Combine seven copies of this estimate with one copy of $Q/a^2 \geq 7\delta/a^2 + 2k + k^2 - 6$, as in that proposition. Their average proves (40). □

That separate substitution would be unjustified: the positive permanent term and its defect must be kept together before the upper chord is used.

Corollary 16.2 (Inherited and joint sufficient domains)**.** *Use the actual child of* $M_{5/4}$ *just defined and suppose* $\xi \geq 3/4$. *Then* $k \geq 11/20$ *implies* $Q(z) \geq a^2/200$. *If instead*

$$k \geq 109/200, \qquad E \geq E_{\text{floor}}, \qquad E_{\text{floor}}/a^2 \geq 1/500, \tag{41}$$

then $\mathcal{D}(z) \geq a^2/1000$. *Valid choices of* E_{floor} *include* 5! *times the right-hand side of* (29)*, with its stated endpoint convention, or the zero-sector floor* $E_{\text{floor}} = 5\mathcal{A}/324$.

Proof. The inherited floor permits $D = D_0 = 181359/1345600$. The polynomial $L_D(k)$ increases in both $k \geq 0$ and $D \geq 0$. Exact substitution gives

$$L_{D_0}(11/20) = \frac{900047936124297}{144482699837440000} > \frac{1}{200},$$
$$L_{D_0}(109/200) = -\frac{4596464533312903}{14448269983744000000}.$$

Adding $1/500$ to the latter gives $24300075434175097/14448269983744000000 > 1/1000$. The zero-sector term of Theorem 13.1, in a containing support space of dimension six, is $120\mathcal{A}/6^5 = 5\mathcal{A}/324$. The containing space may include unused dimensions and excludes no kernel component of z. □

In the second domain a negative *uniform lower bound* for Q is paid by actual variance. Neither actual negativity of Q nor failure of every earlier optimized certificate follows.

The joint domain is nonempty on actual rank-five boundary data. Let $B = J_2 \oplus I_4$, and let $X = \rho J_2 \oplus 0_4$, where $\rho = 109/400$. Realize the two repeated rows by e_1 and the four singletons by $e_2, \dots, e_5$, in ambient dimension six, and choose $z = \sqrt{\rho}\, e_1 + \sqrt{1-\rho}\, e_6$. Then z is unit, $B - X \succeq 0$, $a = 2$, $k = 2\rho = 109/200$, and $\mathcal{A} = 4\rho(1-\rho)$. Deleting a singleton from the normalized $M_{5/4}$ leaves its repeated pair unchanged and gives $\xi = 1$. Moreover

$$\frac{5\mathcal{A}}{324a^2} = \frac{10573}{3456000} > \frac{1}{500}.$$

This example establishes nonemptiness, rather than necessity of the new certificate. The universal results already allow complex phases, rank-five bases, active unit directions and arbitrary kernel mixtures; their numerical hypotheses must be checked on the actual chosen direction.

17 Paired curvature and the remaining occupation budget

The same paired polynomial has a nonnegative remainder that can be kept instead of using convexity alone.

Proposition 17.1 (Finite-forward curvature identity)**.** *For $h \ge 0$ define*

$$R_h = C(1+h) - c - 2hu - h^2\ell.$$

Then $R_h \ge 0$. In particular, with $h = \sqrt{2} - 1$ and $P = a + t$,

$$\mathcal{D}(z) = P^2 - F(\sqrt{2})^2 + \Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E + R_{\sqrt{2}-1}. \tag{42}$$

Consequently

$$\Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E \ge F(\sqrt{2})^2 - P^2 \tag{43}$$

is sufficient for $\mathcal{D}(z) \ge 0$.

Proof. Put $A_0 = |B|^2$, $Y = B \circ \overline{X} + \overline{B} \circ X$, and $W = |X|^2 = pp^{\mathsf{T}}$. These three matrices are PSD by Schur products. Labelled orthogonal Gram sectors, as in Lemma 11.1, give nonnegative coefficients in $\operatorname{per}(A_0 + vY + wW)$. Therefore $C(1+h) = \operatorname{per}(A_0 + hY + h^2W)$ has nonnegative coefficients in h. Its constant and linear terms are c and $2hu$. Its one-W contribution to degree two is $h^2 D \operatorname{per}(A_0)[W] = h^2\ell$; all other contributions are nonnegative. This proves $R_h \ge 0$ without assigning a sign to individual entries or permutation terms. Since $u = \ell + E$,

$$C(1+h) = c + h(2+h)\ell + 2hE + R_h.$$

At $h = \sqrt{2} - 1$ the coefficient $h(2+h)$ equals one. Insert $C(\sqrt{2}) = F(\sqrt{2})^2 - \Delta_{\sqrt{2}}$ into $\mathcal{D} = P^2 - c - \ell$ to obtain (42). Dropping its nonnegative remainder proves the sufficient condition. □

The identity is exact. It isolates an actual-data comparison.

Corollary 17.2 (Top ordinary occupation loss)**.** *Write $F(s) = \sum_{j=0}^{6} \alpha_j s^j$, with $\alpha_j \ge 0$, and put $\alpha_6 = 720 \prod_i p_i$. Then*

$$(F(\sqrt{2})^2 - P^2)_+ \le \alpha_6(2P + \alpha_6).$$

In particular, either of the following conditions certifies $\mathcal{D}(z) \ge 0$:

$$\Delta_{\sqrt{2}} + 2(\sqrt{2}-1)E \ge \alpha_6(2P + \alpha_6), \tag{44}$$

$$t \ge 21\alpha_6. \tag{45}$$

Any possible negative $\mathcal{D}$ not already excluded by the second condition must have $t > 0$ and $6\alpha_6/t > 2/7$.

Proof. Border expansion gives $P = \sum_j (j+1)\alpha_j$. For $j \le 5$, $g_j = j + 1 - 2^{j/2} \ge 0$, while $g_6 = -1$. Thus $P - F(\sqrt{2}) \ge -\alpha_6$. If $F(\sqrt{2}) > P$, set $d = F(\sqrt{2}) - P \le \alpha_6$; the squared loss is $d(2P+d) \le \alpha_6(2P+\alpha_6)$. Otherwise it is nonpositive. This proves the first condition using (43). For $1 \le j \le 4$, $g_j \ge 1/2 \ge j/15$; also $g_5 = 6 - 4\sqrt{2} > 1/3 = 5/15$, as follows from $289 > 288$. Since $t = \sum_j j\alpha_j$, it follows that

$$P - F(\sqrt{2}) \ge \frac{t - 21\alpha_6}{15}.$$

Condition (45) makes the ordinary squared loss nonpositive. If $t = 0$, nonnegative coefficients force $\alpha_6 = 0$, so this endpoint also passes. The final restriction is the contrapositive. □

This filter quantifies the coefficient comparison in the attributed small-order induction. The actual inherited child reserve at $M_{\sqrt{2}}$ and the joint trace-sector lower bound for E may be inserted into (43) or (44). A failure of these stronger sufficient budgets would not by itself imply a negative $\mathcal{D}$, because $R_{\sqrt{2}-1}$ is still available.

18 A limitation of defect inheritance

The forward reserve must be measured on the actual transformed base. It cannot be replaced by a positive fixed multiple of the starting defect.

Proposition 18.1 (One forward point can erase the base defect)**.** *For every fixed $s_0 > 1$, there is a positive definite six-row correlation matrix with a unit Gram border direction for which $\delta > 0$ but $\Delta_{s_0} = 0$. Hence no universal positive constant $c(s_0)$ satisfies $\Delta_{s_0} \ge c(s_0)\delta$. Such a positive constant is impossible even if the inputs are restricted to full-support genuinely complex rank-six bases.*

Proof. On two rows take

$$H = \frac{s_0}{s_0+1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad X = \frac{1}{s_0+1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = H + X,$$

and adjoin I_4 to B, H, and 0_4 to X. The pair has diagonal 1 and off-diagonal $(1 - s_0)/(s_0 + 1)$; both its eigenvalues are positive. The split is realized by first two unit rows $(\sqrt{s_0/(s_0+1)}, 1/\sqrt{s_0+1})$ and $(-\sqrt{s_0/(s_0+1)}, 1/\sqrt{s_0+1})$, four orthogonal singleton rows, and the unit direction selecting the second coordinate. The pencil pair has diagonal $d_s = (s_0 + s)/(s_0 + 1)$ and off-diagonal $e_s = (s - s_0)/(s_0 + 1)$, so direct two-row expansion gives

$$\Delta_s = 2d_s^2 e_s^2 = \frac{2(s_0+s)^2(s-s_0)^2}{(s_0+1)^4}.$$

Thus $\Delta_1 > 0$ and $\Delta_{s_0} = 0$.

For the stronger restriction, use this six-row B and $w = (1, i, 1, 1, 1, 1)^{\mathsf{T}}$. For $\varepsilon > 0$ put

$$B_\varepsilon = \frac{B + \varepsilon w w^*}{1+\varepsilon}, \quad X_\varepsilon = \frac{X}{1+\varepsilon}, \quad H_\varepsilon = \frac{H + \varepsilon w w^*}{1+\varepsilon}.$$

This is still an actual unit Gram-direction split, obtained by appending one coordinate to the rows and keeping the old selected unit direction. The base is positive definite with diagonal one and every entry nonzero. Writing $b = (1 - s_0)/(1 + s_0) < 0$, the imaginary part of its triangle product on rows $1, 2, 3$ is $b\varepsilon^2/(1+\varepsilon)^3 \ne 0$. Its phases cannot all be removed by diagonal unitary gauges. Continuity gives $\Delta_{s_0,\varepsilon}/\delta_\varepsilon \to 0$, while $\delta_\varepsilon \to \delta > 0$. This disproves a positive uniform constant on that restricted class too. □

The perturbed matrices need not have zero forward defect individually; the limit alone disproves a uniform positive transfer. For $s_0 = 5/4$, the exact ordinary polynomial and defect are

$$F(s) = \frac{50 + 32s^2}{81}, \qquad \Delta_s = \frac{2(16s^2 - 25)^2}{6561}.$$

Here $\delta = 2/81$, $E = 3200/6561$, $Q = 8930/6561$ and $\mathcal{D} = 12130/6561 > 0$. Thus the obstruction invalidates the proposed transfer, not Chollet's inequality or the conditional child reserve of Proposition 16.1.

Proposition 18.2 (A qualified two-point reserve)**.** *Let $\eta = \max_{i \neq j} |B_{ij}|^2$, and use the same actual pencil at $s_1 = 5/4$, $s_2 = 3/2$. With $\overline{D}$ from Theorem 11.3,*

$$\max_{j=1,2} \frac{\Delta_{s_j}}{a^2} \geq \overline{D}(\eta/16).$$

Proof. Let η_j be the maximum normalized squared coherence of M_{s_j}. Its diagonal entries are $1 + (s_j - 1)p_i \in [1, s_j]$. Since $B = 2M_{s_1} - M_{s_2}$, every off-diagonal pair satisfies

$$|B_{ik}| \leq 2s_1\sqrt{\eta_1} + s_2\sqrt{\eta_2} \leq 4\max\{\sqrt{\eta_1}, \sqrt{\eta_2}\}.$$

Hence some $\eta_j \geq \eta/16$. Normalize that matrix diagonally and apply Theorem 11.3. The defect-to-squared-permanent ratio is unchanged by normalization, and $F(s_j) \geq a$ by its nonnegative coefficients. Monotonicity of $\overline{D}$ proves the claim. □

The favorable index depends on the pencil. The proposition gives one reserve among two points, not separate lower bounds at both. At $\eta = 0$ it supplies only zero. The factor 16 is conservative and this statement alone does not pay the generic compensation budget. The arguments throughout these sections preserve complex phases and singular limits; no conditioning cutoff is used.

19 Ordinary occupation constrains the actual variance

The ordinary occupation polynomial and the cofactor-weighted variance have the same Gram factors. The following bound retains this relation without a unit-row or small-order hypothesis. The polynomial here is F, not the residual polynomial G_z of Proposition 3.1.

For this section let $n \geq 1$, $w_i = V^* e_i$ be arbitrary vectors and z be unit. Use the conjugate-linear-first inner product and put

$$B_{ij} = \langle w_i, w_j \rangle, \quad x_i = \langle w_i, z \rangle, \quad p_i = |x_i|^2, \quad P_i = w_i w_i^*, \quad r_i = (I - zz^*)P_i z.$$

With the actual transposed cofactor matrix L of $|B|^2$, define $E = \sum_{ij} L_{ij}\langle r_i, r_j \rangle$ as before. Thus $E = z^*Uz - p^{\mathsf{T}}Lp$ also for nonunit rows. Write

$$F(s) = \operatorname{per}(H + sX) = \sum_{j=0}^{n} \alpha_j s^j, \quad H = B - xx^* \geq 0, \quad X = xx^*, \quad \rho = \prod_i p_i.$$

The occupation expansion gives $\alpha_j \geq 0$, $F(1) = a$, $F'(1) = t = z^*Tz$ and $\alpha_n = n!\rho$.

Theorem 19.1 (Sharp occupation–variance coupling)**.** *For arbitrary complex Gram vectors and unit z, in every order $n \geq 1$,*

$$E \geq \rho(na - t) = \frac{\alpha_n}{n!} \sum_{j=0}^{n} (n - j)\alpha_j. \tag{46}$$

The universal coefficient 1 *is optimal in each fixed order $n \geq 2$. At order one the inequality is equality. The statement includes zero occupations, singular factors and all ambient kernel components.*

Proof. If $\rho = 0$, the right side is zero and the cofactor Gram identity gives $E \geq 0$. Suppose $\rho > 0$. Phase gauges leave P_i and E unchanged, so arrange

$$w_i = \sqrt{p_i}(z + y_i), \qquad y_i \perp z.$$

Then $r_i = p_i y_i$. Set $d_i = w_i \otimes \overline{w_i}$ and let q_i be the averaged symmetric tensor product of the $n-1$ doubled vectors other than i; the empty product is the vacuum. Their Gram matrix is $L/(n-1)!$. Consequently

$$Y = \sum_i r_i \otimes q_i, \qquad E = (n-1)!\|Y\|^2.$$

Regroup the doubled slots into ordinary and conjugate legs, a unitary identification. Contract every conjugate omitted leg with $\overline{z}$. On a single doubled factor this contraction gives $p_j(z + y_j)$. It commutes with simultaneous permutation of the paired slots. Thus the contracted tensor is exactly

$$\rho Z, \qquad Z = \sum_i y_i \otimes \mathrm{Sym}_{n-1} \bigotimes_{j \neq i} (z + y_j), \qquad E \geq (n-1)!\rho^2 \|Z\|^2.$$

Put $f = \mathrm{Sym}_n \bigotimes_i (z + y_i)$ and let f_l be its component with exactly l factors in $z^\perp$. Likewise write $Z = \sum_{l=1}^n Z_l$ by total perpendicular occupation, including the distinguished first slot. These are orthogonal components. For full ordinary symmetrization S_n, expansion counts each l-perpendicular monomial l times, giving

$$S_n Z_l = l f_l.$$

On the fully symmetric total-l sector, the first-slot perpendicular projection $P_\perp^{(1)}$ satisfies

$$S_n P_\perp^{(1)} S_n = \frac{l}{n} I.$$

Indeed all n slot projections have equal compression and their sum is the perpendicular number operator, with eigenvalue l. The adjoint output is symmetric in the remaining $n-1$ slots. Hence the restriction of S_n to first-slot-perpendicular tensors in this sector has squared operator norm l/n, and

$$\|Z_l\|^2 \geq \frac{n}{l}\|S_n Z_l\|^2 = nl\|f_l\|^2.$$

The ordinary sector normalization is $\alpha_{n-l} = n!\rho\|f_l\|^2$. Summing gives

$$E \geq n!\rho^2 \sum_{l=1}^n l\|f_l\|^2 = \rho \sum_{j=0}^n (n-j)\alpha_j,$$

proving (46).

For sharpness, take all w_i equal to the same unit vector w and choose unit z with $|\langle w, z\rangle|^2 = q \in (0,1)$. Then $a = n!$, $t = nn!q$ and $E = nn!q(1-q)$, whereas $\rho(na - t) = nn!q^n(1-q)$. Their ratio q^{1-n} tends to one as $q \uparrow 1$, excluding every larger universal coefficient. At $n = 1$, directly $E = p_1(\|w_1\|^2 - p_1)$, which is equality. □

The sharpness family proves an all-Gram coefficient assertion. The ingredients are the classical cofactor tensor and symmetric occupation geometry, here applied to one common omitted residual.

Theorem 19.2 (Occupation-resolved constraint)**.** *Keep the same vectors and scale only their selected mode:*

$$w_i(s) = (I - zz^*)w_i + \sqrt{s}\, zz^* w_i, \qquad s \geq 0.$$

Let L_s be the actual cofactor matrix of $|H + sX|^2$, and let E_s be the variance formed from these scaled vectors and the same unit z. There are nonnegative coefficients β_m such that

$$E_s = \sum_{m=1}^{2n-1} \beta_m s^m, \qquad \beta_{n+j} \geq \rho(n-j)\alpha_j \quad (0 \leq j \leq n-1). \tag{47}$$

Equivalently, $E_s - \rho s^n[nF(s) - sF'(s)]$ has nonnegative coefficients. No diagonal normalization of the scaled vectors is made.

Proof. The residual of the scaled row is $r_i(s) = \sqrt{s}\, r_i$. In the tensor Y above, let $Y_{k,l}$ be its orthogonal component with k selected occupations in the ordinary omitted legs and l in their conjugate legs, where $0 \le k, l \le n-1$. The distinguished residual slot is perpendicular and is not counted in k. The common mode scaling gives the exact identity

$$E_s = (n-1)! \sum_{k,l=0}^{n-1} s^{k+l+1} \|Y_{k,l}\|^2.$$

Only orthogonality of the two number decompositions is used; independent permutation invariance of Y is not assumed. This proves nonnegativity of every β_m.

For $\rho > 0$, the fully occupied conjugate component with ordinary occupation j is exactly ρZ_{n-j} under contraction with $\overline{z}^{\otimes(n-1)}$. The restricted symmetrization estimate in the preceding proof therefore gives

$$\begin{aligned} \beta_{n+j} &\ge (n-1)! \|Y_{j,n-1}\|^2 \\ &\ge n! \rho^2 (n-j) \|f_{n-j}\|^2 = \rho(n-j)\alpha_j. \end{aligned}$$

For $\rho = 0$ the claimed lower bounds are zero. This proves the coefficient statement, including all zero-occupation boundaries. □

In particular a two-endpoint ordinary polynomial still forces $\beta_n \ge n\rho\alpha_0$. These are constraints on jointly realizable ordinary and paired-variance occupations. Merely optimizing E_s separately does not improve (46): for $s \ge 1$ use $E_s \le s^{2n-1} E_1$, and for $0 < s \le 1$ use $E_s \le s E_1$. Combining either bound with Theorem 19.2 gives a lower bound for E_1 maximized at $s = 1$. Further use must retain these aligned coefficients with their actual forward defect.

19.1 Comparison with the trace-sector reserve

At six rows (46) gives

$$E \ge \frac{\alpha_6}{720}(6a - t).$$

This can improve the named trace-sector lower cost substantially. Take orthonormal vectors $z, e_1, \dots, e_6$ and

$$w_i = \sqrt{q}\, z + \sqrt{1-q}\, e_i, \qquad q = 9/10.$$

Then $B = (1-q)I_6 + qJ_6$ has rank six and $\alpha_j = \binom{6}{j} j! q^j (1-q)^{6-j}$. Exact values are

$$a = \frac{17104189}{40000}, \quad t = \frac{1259058357}{500000}, \quad \rho(6a - t) = \frac{6312407836869}{250000000000}.$$

For these actual data, the trace variance and frame variance are both $6q(1-q) = 27/50$, the support dimension is six and $\pi = 6q/(1+5q) = 54/55$. Substituting these quantities into 5! times the right-hand side of (29) gives exactly $83/9930$. Thus the new floor exceeds 25, whereas that named cost is less than $1/100$. This compares specific valid lower bounds on the same data, not all earlier methods or newly covered Chollet instances. The bounds are generally incomparable: if $\rho = 0$ the new floor can vanish while the earlier trace-variance floor is positive. Their maximum is always a valid lower bound; their sum requires a separate proof that the charged components do not overlap.

Nor does the new reserve alone close the forward budget. For the unit binary six-root family $w_i = (e_0 + \omega^i e_1)/\sqrt{2}$ ($0 \le i \le 5$, $\omega^6 = 1$ primitive), with selected mode $z = e_1$,

$$F(s) = \frac{45}{4}(1 + s^6), \quad a + t = 90, \quad F(\sqrt{2}) = 405/4.$$

The positive ordinary squared loss is $34425/16$, while the new variance floor is only $135/128$. Multiplying that floor by $2(\sqrt{2} - 1)$ gives less than two. This failure of a lower-bound certificate is not failure of the actual finite-forward budget or a counterexample to Chollet; the actual transformed defect remains available.

20 The stronger forward budget reduces to active directions

Return to six unit Gram rows and the finite-forward remainder of Proposition 17.1. Put $h = \sqrt{2} - 1$ and

$$J(z) = (a+t)^2 - C_z(1+h) + 2hE(z) = \mathcal{D}(z) - R_h(z), \qquad C_z(1+s) = \operatorname{per}|B + s(Vz)(Vz)^*|^2.$$

Thus $J \geq 0$ is the sufficient budget (43), with the nonnegative remainder discarded. Let $\mathcal{H}_{\mathrm{act}} = (\ker V)^\perp$.

Proposition 20.1 (Curvature credit under kernel thinning)**.** *For active unit w, kernel unit κ, $0 \leq q \leq 1$ and arbitrary real θ, put $z_q = \sqrt{q}\, w + e^{i\theta}\sqrt{1-q}\,\kappa$. There are actual nonnegative coefficients $r_2, \ldots, r_{12}$, independent of q, θ, such that*

$$R_h(z_q) = \sum_{m=2}^{12} r_m q^m, \tag{48}$$

$$J(z_q) = (1-q^2)\delta + q^2 J(w) + 2at_w q(1-q) + \sum_{m=3}^{12} r_m(q^2 - q^m), \tag{49}$$

where $t_w = w^*Tw$. Consequently $J(w) \geq 0$ implies $J(z_q) \geq 0$. For $q > 0$, the function $J(z_q)/q^2$ is nonincreasing in q.

Proof. Since $V\kappa = 0$, $X_q = qX_w, t_q = qt_w, u_q = qu_w, \ell_q = q^2\ell_w$ and $E_q = qu_w - q^2\ell_w$. The PSD-pencil proof of Proposition 17.1 gives

$$C_w(1+s) = \sum_{m=0}^{12} c_m s^m, \qquad c_m \geq 0, \quad c_0 = c, \quad c_1 = 2u_w, \quad c_2 \geq \ell_w.$$

Because $C_{z_q}(1+h) = C_w(1+hq)$, (48) holds with $r_2 = h^2(c_2 - \ell_w)$ and $r_m = h^m c_m$ for $m \geq 3$. Subtract it from $\mathcal{D}(z_q) = \delta + 2at_w q + (t_w^2 - \ell_w)q^2$ to obtain (49). All extra terms are nonnegative, using $\delta \geq 0$ from Lemma 11.1 and $t_w \geq 0$. Furthermore

$$\frac{J(z_q)}{q^2} = \frac{\delta}{q^2} + \frac{2at_w}{q} + t_w^2 - \ell_w - \sum_{m=2}^{12} r_m q^{m-2},$$

whose derivative is $-2\delta/q^3 - 2at_w/q^2 - \sum_{m=3}^{12}(m-2)r_m q^{m-3} \leq 0$. The endpoints $q = 0, 1$ follow directly from the exact identity. □

Every unit vector decomposes by its active projection with weight $q = \|P_{\mathcal{H}_{\mathrm{act}}} z\|^2$; the $q = 0$ budget is $J = \delta \geq 0$. Hence nonnegativity of J on all active unit directions is equivalent to its nonnegativity in every ambient kernel enlargement, including every positive border Schur slack. A negative mixture would force a negative active endpoint. On rank-six bases that endpoint has $\operatorname{rank}(B - X_w) = 5$; on rank-five bases it has rank four. This removes the mixture parameter, without proving the remaining sign.

For completeness, there is an actual reconstruction chart for the remaining active problem. If every x_i is nonzero, set $D = \operatorname{diag}(1/x_i)$. Then $DXD^* = J_6$ and $K = D(B - X)D^* \geq 0$. All terms in J scale by the same positive factor $|\det D|^4$: a, t scale by $|\det D|^2$, and c, u, ℓ, E by $|\det D|^4$. Thus the sign is unchanged. Conversely, any $K \geq 0$ with $X = J_6$ is realized by Gram rows $(y_i, 1)$ and the unit border selecting the final coordinate; dividing row i by $\sqrt{1 + K_{ii}}$ restores diagonal one. To specify active status require $\operatorname{rank}(K + J_6) = \operatorname{rank} K + 1$: the rank-six active chart has ranks $\operatorname{rank} K = 5$, $\operatorname{rank}(K + J_6) = 6$, and the rank-five chart has ranks four and five. The nonzero-occupation locus is dense in the active unit sphere, being the complement of finitely many proper complex hyperplanes, and J is continuous. A proof on all of this actual chart would therefore include zero occupations by continuity. The all-order variance constraint and this active reduction preserve complex phases and singular boundaries. They do not settle the generic active rank-six budget or the unrestricted Chollet conjecture.

21 Projection overlap couples trace and occupation

Here the rows w_i are unit, $n \geq 2$, and z is any ambient unit vector. Let S be an r-dimensional containing support of the rows, $m = n - 1$, $\pi = \|P_S z\|^2$, $P_i = w_i w_i^*$, and $r_i = (I - zz^*)P_i z$. On the omitted Hilbert–Schmidt factors put

$$Y = \sum_i r_i \otimes \mathrm{Sym}_m \bigotimes_{j \neq i} P_j, \qquad E = m! \|Y\|^2.$$

Symmetrization permutes the paired factors simultaneously. Put $e = I_S/\sqrt{r}$. Let $\mathcal{P}_k$ project the omitted factors onto sectors with at most k traceless slots and all remaining slots equal to e, for $0 \leq k \leq m$. Let Q project every conjugate omitted leg onto $\overline{z}$. Both leave the distinguished residual untouched.

Theorem 21.1 (Exact overlap and safe envelope)**.** *The overlap is*

$$\|\mathcal{P}_k Q\|^2 = \gamma_k^2 = \left(\frac{\pi}{r}\right)^m \sum_{j=0}^{k} \binom{m}{j} (r-1)^j. \tag{50}$$

If $A \leq m!\|\mathcal{P}_k Y\|^2$ and $M \leq m!\|QY\|^2$ are nonnegative proved floors, then $E \geq \mathcal{H}_{\gamma_k}(A, M)$, where for $0 < \gamma < 1$

$$\mathcal{H}_\gamma(A, M) = \begin{cases} \max(A, M), & \min(A, M) \leq \gamma^2 \max(A, M), \\ (A + M - 2\gamma\sqrt{AM})/(1 - \gamma^2), & \textit{otherwise}. \end{cases} \tag{51}$$

The endpoint conventions are $\mathcal{H}_0 = A + M$ and $\mathcal{H}_1 = \max(A, M)$.

Proof. If $\pi = 0$, the overlap is zero. Otherwise set $v = P_S z/\sqrt{\pi}$. After identifying the selected conjugate output with a scalar, compression of one trace projector is $A_0 = |P_S z\rangle\langle P_S z|/r$ on the ordinary slot. Compression of the Hilbert–Schmidt identity supported on S is πI_S, so the compressed traceless projector is $\pi I_S - A_0$. Consequently $Q\mathcal{P}_k Q$ is the sum of commuting tensor products with at most k copies of $\pi I_S - A_0$, the other copies being A_0. On a sector with l ordinary slots perpendicular to v, its eigenvalue is zero if $l > k$, and otherwise is

$$\frac{\pi^m}{r^{m-l}} \sum_{b=0}^{k-l} \binom{m-l}{b} (r-1)^b.$$

Writing $B(d, h) = \sum_{b=0}^{h} \binom{d}{b}(r-1)^b$, the identity $B(d, h) - rB(d-1, h-1) = \binom{d-1}{h}(r-1)^h \geq 0$ compares consecutive l and places the maximum at $l = 0$. The all-v tensor attains it and is simultaneously symmetric. Thus the symmetry restriction and the residual identity do not change the norm. At $r = 1$ the traceless subspace is zero and the same formula holds with $0^0 = 1$.

For arbitrary orthogonal projections P, Q with $\|PQ\| \leq \gamma$, the Gram matrix of the synthesis map from $\operatorname{ran} P \oplus \operatorname{ran} Q$ gives, for $c, d \geq 0$,

$$\|cP + dQ\| \leq \frac{c + d + \sqrt{(c-d)^2 + 4cd\gamma^2}}{2}.$$

In the second branch of (51), choose $c = (1 - \gamma\sqrt{M/A})/(1 - \gamma^2)$ and $d = (1 - \gamma\sqrt{A/M})/(1 - \gamma^2)$. Both are positive, the displayed upper bound is one, and $cA + dM$ is the asserted expression. In the first branch one projection alone suffices. Zero floors and endpoints follow directly. This is classical two-projection geometry; (50) supplies its actual common-Gram overlap. □

Theorem 19.1 gives $M = \rho(na - t)$. Theorem 13.1 supplies a floor A supported entirely in $\mathcal{P}_1 Y$, namely $m!$ times the right side of (29), with its stated zero-denominator convention. Thus these floors can be combined using $\gamma_1^2 = (\pi/r)^m[1 + m(r-1)]$. For six rows and active z, this is $26/7776$ at $r = 6$ and $21/3125$ at $r = 5$. The zero-trace floor $A = m!\mathcal{A}/r^m$ instead uses $\gamma_0^2 = (\pi/r)^m$. One may take the maximum of valid choices. One may not add the floors without paying their overlap, nor apply the second branch of (51) outside its stated range. Higher k require a proved floor in that actual sector.

22 Cofactor occupations and sharp retained curvature

For this section the rows need not be unit. Keep $H = B - X \succeq 0$, $X = xx^*$, $p_i = |x_i|^2$, $\rho = \prod_i p_i$, and write

$$F(v) = \operatorname{per}(H + vX) = \sum_{k=0}^{n} \alpha_k v^k, \quad C(v) = \operatorname{per}|H + vX|^2, \quad W = pp^{\mathsf{T}}.$$

Define

$$N(v) = D\operatorname{per}(|H + vX|^2)[W] = \sum_{k=0}^{2n-2} \nu_k v^k.$$

For the actual scaled rows $w_i(v) = (I - zz^*)w_i + \sqrt{v}\, zz^* w_i$, without renormalizing them, let E_v be their variance. Their selected occupations are vp_i. Put $c = C(1)$, $\ell = N(1)$, $u = C'(1)/2$, and, for $s = 1+h \geq 1$, $R_h = C(s) - c - 2hu - h^2\ell$.

Theorem 22.1 (Cofactor occupation constraints)**.** *All ν_k are nonnegative. For $n \geq 2$,*

$$\nu_{n-2+k} \geq 2\rho k\alpha_k \quad (1 \leq k < n), \tag{52}$$

$$\nu_{2n-2} = \rho n\alpha_n = nn!\rho^2, \qquad \nu_{2n-3} = 2\rho(n-1)\alpha_{n-1}. \tag{53}$$

For $n = 1$, $N(v) = \rho^2$. These statements retain all complex phases, zero occupations and singular factors.

Proof. The pencil $|H+vX|^2 = A_0+vA_1+v^2W$ has PSD coefficients $A_0 = H\circ\overline{H}$ and $A_1 = H\circ\overline{X}+\overline{H}\circ X$, by the Schur product theorem. Expanding the mixed permanent in orthogonally labeled Gram sectors gives nonnegative coefficients, also after taking the derivative in the PSD direction W. This proves $\nu_k \geq 0$.

Suppose first $\rho > 0$ and gauge $w_i = \sqrt{p_i}(z + y_i)$, $y_i \perp z$. Set

$$d_i(v) = (y_i + \sqrt{v}z) \otimes (\overline{y}_i + \sqrt{v}\,\overline{z}), \qquad Z(v) = \sum_i \operatorname{Sym}_{n-1} \bigotimes_{j\neq i} d_j(v).$$

The omitted-tensor Gram identity gives $N(v) = (n-1)!\rho^2\|Z(v)\|^2$. The ordinary/conjugate selected-occupation sectors (a, b) are orthogonal, and contribute to degree $a + b$. Let $f = \operatorname{Sym}_n \bigotimes_i (z + y_i)$ and let f_k have k selected factors. Then $\alpha_k = n!\rho\|f_k\|^2$. The sum of ordinary omitted products is $g = \sum_i \operatorname{Sym}_{n-1} \bigotimes_{j\neq i}(z + y_j) = n\operatorname{contr}_z f$. On the symmetric k-selected sector, first-slot selected projection compresses to k/n; hence $\|g_{k-1}\|^2 = nk\|f_k\|^2$. The $(k-1, n-1)$ sector of Z is exactly g_{k-1}, and contributes $\rho k\alpha_k$ to degree $n-2+k$. Its swapped conjugate sector has the same norm and is distinct for $k < n$. This gives (52). At $k = n$ the sectors coincide, giving the first identity in (53); at total degree $2n-3$ only the two displayed sectors exist, giving the second identity. If $\rho = 0$, inequalities follow from coefficient positivity, while the two identities follow by polynomial continuity. The order-one case is direct. □

Theorem 22.2 (Integral reserve and sharp top credit)**.** *For every $s \geq 1$,*

$$C(s) - c - (s^2 - 1)\ell = 2\int_1^s \frac{E_v}{v}\, dv + 2\int_1^s v[N(v) - N(1)]\, dv = 2hE + R_h. \tag{54}$$

Moreover

$$R_h \geq \frac{\alpha_n^2}{n!}[s^{2n} - 1 - n(s^2 - 1)]. \tag{55}$$

The coefficient is optimal over arbitrary Gram data. For six unit rows and $s = \sqrt{2}$, $R_h \geq (19/240)\alpha_6^2$; the constant remains optimal on active rank-six and active rank-five bases.

Proof. Differentiation of the actual Gram pencil gives $E_v = vC'(v)/2 - v^2N(v)$, because the actual cofactor term is $v^2N(v)$. Integrating $C'(v) = 2E_v/v + 2vN(v)$ proves the first equality in (54); $u = E + \ell$

proves the second. By Theorem 19.2, $E_v = \sum_{j\ge1} \beta_j v^j$ with $\beta_j \ge 0$, so $E_v/v \ge E$ for $v \ge 1$. Subtract $2hE$ in (54) and retain only the top coefficient of N, given by (53). This gives

$$R_h \ge nn!\rho^2 \left[\frac{s^{2n}-1}{n} - (s^2-1) \right],$$

which is (55). The bracket is nonnegative by Bernoulli's inequality. Equality holds when $H = 0$: then $E_v = 0$ and $N(v) = nn!\rho^2 v^{2n-2}$. This includes the zero coefficient at $n = 1$ or $h = 0$.

For the rank-six assertion put $\Pi = I_6 - J_6/6$ and, for $0 < \epsilon < 1$,

$$H_\epsilon = \epsilon\Pi, \quad X_\epsilon = (1 - 5\epsilon/6)J_6, \quad B_\epsilon = \epsilon I_6 + (1-\epsilon)J_6.$$

The residual/base ranks are $5/6$, the diagonal is one and the selected direction is active. As $\epsilon \downarrow 0$, $H \to 0$, $X \to J_6$ and $\alpha_6 \to 720$, so polynomial continuity makes $R_h/\alpha_6^2 \to 19/240$. For rank five take $t = (1,1,1,-1,-1,-1)^{\mathsf{T}}$, $\Pi_5 = I_6 - (J_6 + tt^*)/6$, $H_\epsilon = \epsilon\Pi_5$ and $X_\epsilon = (1-2\epsilon/3)J_6$. These have diagonal one and ranks $4/5$, with the same limiting ratio. One can retain genuinely complex, noncirculant examples by replacing H_ϵ in either family by $\epsilon\Pi_* + \epsilon^2\Pi_* i(E_{12} - E_{21})\Pi_*$. The perturbation has zero diagonal, annihilates the same kernel and has operator norm at most ϵ^2, so positivity and ranks persist. Writing $b = 1-\epsilon$ and $\delta = \epsilon^2$, the imaginary part of $B_{12}B_{23}B_{31}$ is $b^2\delta - \delta^3/54$ for Π, and $b^2\delta - \delta^3/27$ for Π_5. It is nonzero for sufficiently small ϵ, excluding a gauge-real matrix. The ratio limit is unchanged. The sharpness assertion permits complex phases and the stated conditioning range. □

Here is an occupation-resolved replacement for the reserve. Define

$$w_j(s) = 2(s^j - 1)/j, \quad T_j(s) = 2\left[\frac{s^{j+2}-1}{j+2} - \frac{s^2-1}{2} \right].$$

Let $O \ge \rho(na - t)$ be any proved floor for E, for example a maximum of the overlap envelopes above and the occupation floor. Then

$$\begin{aligned} 2hE + R_h \ge 2hO &+ \sum_{j=0}^{n-1} [w_{n+j}(s) - 2h]\rho(n-j)\alpha_j \\ &+ \sum_{k=1}^{n-1} 2\rho k\alpha_k T_{n-2+k}(s) + \rho n\alpha_n T_{2n-2}(s). \end{aligned} \tag{56}$$

Indeed Theorem 19.2 reserves mandatory masses $\beta_{n+j} \ge \rho(n-j)\alpha_j$; every remaining coefficient has weight at least $w_1 = 2h$ in $2\int E_v/v$. The second integral in (54) equals $\sum_k \nu_k T_k(s)$, and (52)–(53) give its displayed floor. These are two separate integrals, so their credits add. The result replaces $2hE + R_h$; it must not be added to that exact quantity again. In particular it cannot repair a failed exact-curvature certificate with the same variance floor simply by counting curvature twice.

23 A uniform boundary band and an order-seven consequence

Use six unit Gram rows, any ambient unit z, and the preceding pencil. Put $P = F(1) + F'(1) = a + t$, $s = \sqrt{2}$, $h = s - 1$, and

$$G = P^2 - c - \ell, \qquad \Delta_s = F(s)^2 - C(s).$$

Lemma 11.1 gives $\Delta_s \ge 0$. The exact compensation identity is

$$G = P^2 - F(s)^2 + \Delta_s + 2hE + R_h = P^2 - C(s) + 2hE + R_h. \tag{57}$$

Theorem 23.1 (Uniform small-coordinate compensation)**.** *If* $\min_i p_i \le 1/401$*, then* $G \ge 0$*. On the positive-loss part* $F(\sqrt{2}) > P$ *of this band,*

$$G > (1-\kappa)P^2 + 2hE + R_h, \qquad \kappa = \frac{(401/232)^{12}}{720} < 1.$$

On the nested band $\min_i p_i \le 1/626$*, positive loss gives the stronger* $G > (2/3)P^2 + 2hE + R_h$*. No phase, rank or conditioning restriction is imposed.*

Proof. If $F(s) \le P$, (57) proves the assertion. If some $p_i = 0$, then $\alpha_6 = 0$, and $j+1 \ge 2^{j/2}$ for $0 \le j \le 5$ implies $F(s) \le P$. We may therefore assume positive loss and all $p_i > 0$. Gauge $w_i = (z+y_i)/\sqrt{1+\|y_i\|^2}$ with $y_i \perp z$ and $p_i = (1+\|y_i\|^2)^{-1}$. For $e_l(y) = \sum_{|I|=l} \mathrm{Sym}_l \bigotimes_{i\in I} y_i$, orthogonality of the selected placements gives

$$\alpha_{6-l} = \frac{720\rho}{\binom{6}{l}} \|e_l(y)\|^2.$$

Put $g_j = j + 1 - 2^{j/2}$. Then $g_0 = 0$, $g_6 = -1$, and $g_j > 0$ for $1 \le j \le 5$. Positive loss is $\alpha_6 > \sum_{j=1}^{5} g_j \alpha_j$. Thus, for $l = 1, \dots, 5$, the respective upper bounds on $\|e_l(y)\|^2$ are

$$\frac{6}{6-4\sqrt{2}}, \quad 15, \quad 5(2+\sqrt{2}), \quad 15, \quad 3(2+\sqrt{2}).$$

All are less than 18: the tight first comparison reduces to $17 > 12\sqrt{2}$, whose square is $289 > 288$. Consequently $\|e_l(y)\| < 9/2$ for these five degrees.

Let $R_0 = \max_i \|y_i\|$ and choose a unit v along a maximizing row. Set $x_i = \langle v, y_i\rangle$. One $x_i = R_0$ and all $|x_i| \le R_0$. Contraction against $v^{\otimes l}$ gives $|e_l(x)| < 9/2$. Evaluating $\prod_i (T - x_i)$ at the actual root R_0 yields, for $R_0 > 1$,

$$\left|\prod_i x_i\right| > R_0^6 \left[1 - \frac{9/2}{R_0 - 1}\right].$$

This uses only a finite geometric sum and holds for arbitrary complex roots. On the full band $R_0 \ge 20$, the bracket is at least $29/38$. Every $|x_i|$ therefore exceeds $29R_0/38$, since all the other factors are at most R_0. It follows that

$$p_i < \frac{1444}{1444 + 841R_0^2} \le \frac{361}{84461} < \frac{1}{200}, \qquad q_i := |\langle v, w_i\rangle|^2 > \frac{841R_0^2}{1444(1+R_0^2)} \ge \frac{84100}{144761} > \frac{29}{50}.$$

Projection of the actual symmetric Gram product onto $v^{\otimes 6}$ gives $a \ge 720 \prod_i q_i > 720(29/50)^6$. For $M = B + hX \ge 0$, the diagonal satisfies $M_{ii} = 1 + hp_i < 401/400$, since $h < 1/2$. Its two-by-two PSD minors give $|M_{ij}| < 401/400$. The classical Carlen–Lieb–Loss inequality [18], applied to $|M|^2$, gives

$$C(s) \le U_{\mathrm{CLL}}(M) := \frac{720}{216} \prod_i \sqrt{\sum_j |M_{ij}|^4} < 720(401/400)^{12}.$$

Since $P \ge a$, $C(s)/P^2 < \kappa < 1$; the last comparison is the exact integer inequality $401^{12} < 720\,232^{12}$. Equation (57) proves the stated margin.

On the nested band $R_0 \ge 25$, the same bracket is at least $13/16$. Thus $p_i < 256/105881 < 1/400$ and $q_i > 105625/160256 > 13/20$. Now $M_{ii} < 801/800$ and

$$C(s)/P^2 < (801/520)^{12}/720 < 1/3,$$

where the last comparison is $801^{12} < 240\,520^{12}$. This proves the stronger margin. No inverse or lower eigenvalue bound entered either argument. □

23.1 Retaining the joint coefficient budget and coherent product

The preceding $1/401$ band can be enlarged by retaining the joint coefficient constraint and the product of the coherent projections. The stronger $1/626$ margin in Theorem 23.1 remains useful.

Theorem 23.2 (Joint root-product compensation)**.** *For six unit complex Gram rows and any ambient unit z,*

$$\min_i p_i \le \frac{1}{31} \quad \Longrightarrow \quad G \ge 0.$$

On the positive-loss part $F(\sqrt{2}) > P$ of this band,

$$C(\sqrt{2}) < \kappa_{31} P^2, \qquad G > \frac{P^2}{20} + 2hE + R_h, \tag{58}$$

where

$$\kappa_{31} = \left(\frac{377}{360}\right)^{12} \frac{40^4}{720\,9^4} = \frac{8243206936713178643875538610721}{87436462389547861278720000000000} < \frac{19}{20}.$$

All ranks, complex phases, singular limits and ambient kernel components are permitted.

Proof. If $F(s) \le P$, identity (57), with $\Delta_s, E, R_h \ge 0$, proves $G \ge 0$. If some $p_i = 0$, then $\alpha_6 = 0$ and $j + 1 \ge 2^{j/2}$ for $0 \le j \le 5$ forces $F(s) \le P$. We may therefore assume positive loss and all $p_i > 0$. As before, gauge

$$w_i = \sqrt{p_i}(z + y_i), \quad y_i \perp z, \quad p_i = (1 + \|y_i\|^2)^{-1}, \quad \rho = \prod_i p_i.$$

For $e_l(y) = \sum_{|I|=l} \mathrm{Sym}_l \bigotimes_{i\in I} y_i$, the exact ordinary sector identity gives $\alpha_{6-l} = 720\rho\|e_l(y)\|^2/\binom{6}{l}$. Put $g_j = j + 1 - 2^{j/2}$. Since $g_0 = 0$, $g_6 = -1$ and $g_1, \dots, g_5 > 0$, positive loss gives the joint bound

$$\sum_{l=1}^{5} \frac{g_{6-l}}{\binom{6}{l}} \|e_l(y)\|^2 < 1. \tag{59}$$

Let $R_0 = \max_i \|y_i\|$ and choose a unit v along a maximizing row. The hypothesis implies $R_0^2 \ge 30$. Define $\xi_i = \langle v, y_i\rangle$; one $\xi_i = R_0$ and every $|\xi_i| \le R_0$. Contracting against the unit tensors $v^{\otimes l}$ preserves (59) with $e_l(\xi)$ in place of $e_l(y)$. Evaluate $\prod_i (T - \xi_i)$ at its root R_0. The triangle inequality and weighted Cauchy–Schwarz give

$$\begin{aligned} \left|\prod_i \xi_i\right| &\ge R_0^6 - \sum_{l=1}^{5} |e_l(\xi)| R_0^{6-l} \\ &> R_0^6 \left[1 - \sqrt{S(R_0^2)}\right], \end{aligned} \tag{60}$$

where the exact reciprocal weights yield

$$S(q) = \frac{9 + 6s}{q} + \frac{15}{q^2} + \frac{10 + 5s}{q^3} + \frac{15}{q^4} + \frac{6 + 3s}{q^5}, \qquad s = \sqrt{2}.$$

All coefficients are positive, so S decreases on $q > 0$. Using $s < 99/70$, justified by $9800 < 9801$, gives

$$S(R_0^2) \le S(30) < \frac{48614177}{81000000} = \frac{961}{1600} - \frac{1139}{2531250} < \left(\frac{31}{40}\right)^2.$$

Thus (60) gives $|\prod_i \xi_i| > (9/40)R_0^6$. This argument applies to complex scalar roots without any real-rootedness or stability assumption.

Projection of the actual symmetrized six-row product onto $v^{\otimes 6}$ now gives the product bound

$$a \geq 720 \prod_i |\langle v, w_i\rangle|^2 = 720\rho \left|\prod_i \xi_i\right|^2 > 720\rho(9/40)^2 R_0^{12}.$$

For the actual PSD forward matrix $M = H + sX$, every two-by-two PSD minor gives $|M_{ij}|^2 \leq M_{ii}M_{jj}$. Each of the 720 permutation terms of $\operatorname{per}|M|^2$ is therefore at most $\prod_i M_{ii}^2$. Since $M_{ii} = p_i(s + \|y_i\|^2)$,

$$C(s) \leq 720 \prod_i M_{ii}^2 = 720\rho^2 \prod_i (s + \|y_i\|^2)^2 \leq 720\rho^2 (s + R_0^2)^{12}.$$

Combining this elementary PSD bound with $P \geq a$ yields

$$\frac{C(s)}{P^2} < \frac{(1 + s/R_0^2)^{12}}{720(9/40)^4} \leq \frac{(1 + s/30)^{12}}{720(9/40)^4} < \left(\frac{377}{360}\right)^{12} \frac{40^4}{720\,9^4} = \kappa_{31} < \frac{19}{20}.$$

Here $s < 17/12$ is used only for $1 + s/30 < 377/360$; it is too coarse for the earlier estimate of $S(30)$. For the displayed reduced numerator N and denominator D of κ_{31}, the final comparison is the positive integer identity

$$19D - 20N = 1265139805877363552057227785580 > 0.$$

Identity (57) proves (58). The proof uses the actual common factors and an elementary PSD upper bound; it does not require a generic capacity bridge or a lower Gram eigenvalue. □

Corollary 23.3 (One weak pair at order seven)**.** *Let $A \geq 0$ have order seven and positive diagonal. If some $i \neq j$ satisfies $|A_{ij}|^2/(A_{ii}A_{jj}) \leq 1/31$, then*

$$\operatorname{per}|A|^2 \leq (\operatorname{per} A)^2.$$

If both order-seven PSD matrices A, D have such a pair, possibly different pairs, then $\operatorname{per}(A \circ D) \leq \operatorname{per} A \operatorname{per} D$. *A zero diagonal in either factor is a trivial zero-row endpoint.*

Proof. Positive diagonal congruence normalizes A to a correlation matrix and scales both sides equally. Realize it by seven unit vectors and use the j endpoint of the weak pair as the border z. The other six rows have one occupation at most $1/31$. Border expansion gives exactly $\operatorname{per} A = a + t$ and $\operatorname{per}|A|^2 = c + \ell$, so Theorem 23.2 applies. An ambient kernel component in z is allowed, so this includes full-rank order seven. For two factors, permutation Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ D)| \leq \sqrt{\operatorname{per}|A|^2 \operatorname{per}|D|^2} \leq \operatorname{per} A \operatorname{per} D.$$

The Schur product is PSD and its permanent nonnegative. Each self-bound was used, so both factors must satisfy the stated sufficient condition. A zero diagonal in a PSD matrix forces a zero row and column. □

Any unresolved positive-loss six-row instance must have every $p_i > 1/31$. In the gauge-normalized residual chart $K_{ij} = H_{ij}/\sqrt{p_i p_j}$, this gives $0 \leq K_{ii} < 30$. The closure $K \geq 0$, $K_{ii} \leq 30$ is compact by the PSD entry bound. Its singular boundaries remain included. In particular, a hypothetical order-seven self-conjugate counterexample must have every normalized off-diagonal squared modulus strictly greater than $1/31$.

24 Balanced capacity and coherent-column domains

These are further sufficient conditions on actual six-row pencils, now allowing nonunit rows. Set $M = H + \sqrt{2}X$, $Q = |M|^4$ entrywise, and $P = F(1) + F'(1)$. Suppose M has positive diagonal; a zero diagonal is already a zero-row permanent endpoint. Classical CLL and matrix scaling [18, 19] give the following useful normalization.

Proposition 24.1 (Weighted CLL and symmetric scaling)**.** *For positive v,*

$$C(\sqrt{2}) \le \frac{10}{3}\sqrt{\frac{\prod_i (Qv)_i}{\prod_j v_j}}, \qquad U_{\mathrm{opt}}(M) := \frac{10}{3}\sqrt{\operatorname{cap} Q}, \quad \operatorname{cap} Q = \inf_{v>0} \frac{\prod_i (Qv)_i}{\prod_j v_j}.$$

There is a unique positive d with $d_i(Qd)_i = 1$, and $U_{\mathrm{opt}}(M) = (10/3)/\prod_i d_i$. Congruence $M \mapsto DMD^*$ scales U_{opt} and P^2 by the same factor $\prod_i |D_{ii}|^4$.

Proof. Apply CLL to $|M|^2 \operatorname{diag}(\sqrt{v})$ and divide its permanent by $\prod_j \sqrt{v_j}$ to get the first inequality. For existence and uniqueness of d, minimize

$$f(x) = \frac{1}{2}\sum_{ij} Q_{ij} e^{x_i + x_j} - \sum_i x_i.$$

Positive diagonal makes its Hessian positive definite: diagonal terms contribute $2Q_{ii}e^{2x_i}$, and other terms contribute positive multiples of $(e_i+e_j)(e_i+e_j)^{\mathsf{T}}$. Furthermore $f(x) \ge \sum_i [(Q_{ii}/2)e^{2x_i} - x_i]$ is coercive, since every summand is bounded below and diverges at either endpoint. The unique minimizer satisfies $e^{x_i}\sum_j Q_{ij}e^{x_j} = 1$. Thus $\widehat{Q} = \operatorname{diag}(d) Q \operatorname{diag}(d)$ is doubly stochastic, including when Q is reducible. Weighted AM–GM gives $\prod_i (\widehat{Q}v)_i \ge \prod_j v_j$, with equality at $v = \mathbf{1}$. So $\operatorname{cap}\widehat{Q} = 1$ and $\operatorname{cap} Q = \prod_i d_i^{-2}$. Finally $Q \mapsto RQR$, $R = \operatorname{diag}(|D_{ii}|^4)$, changes capacity by $\prod_i |D_{ii}|^8$, proving covariance. □

Scale the actual rows by $d_i^{1/4}$, so that Q henceforth is doubly stochastic and $U_{\mathrm{opt}} = 10/3$. This does not preserve unit row norms; $F(v)$ and P scale by $\prod_i \sqrt{d_i}$, while G, E, R_h, C scale by $\prod_i d_i$. All margins below are in these balanced units.

Theorem 24.2 (Balanced coherent columns)**.** *If, for some column j,*

$$119070 \prod_i Q_{ij} \ge Q_{jj}^3, \tag{61}$$

then $P^2 \ge U_{\mathrm{opt}} \ge C(\sqrt{2})$ and $G \ge 2hE + R_h \ge 0$. If every $Q_{ij} \ge 1/9$, then more precisely

$$F(\sqrt{2}) \ge 10/3, \quad P \ge 35/12, \quad G \ge 745/144 + 2hE + R_h, \qquad U_{\mathrm{opt}}/P^2 \le 96/245.$$

Proof. Let u_i be actual Gram vectors for M and choose $v = u_j/\|u_j\|$. Projection onto $v^{\otimes 6}$ gives

$$F(\sqrt{2}) = \operatorname{per} M \ge 720 \prod_i |\langle v, u_i\rangle|^2 = 720 \frac{\sqrt{\prod_i Q_{ij}}}{Q_{jj}^{3/2}}.$$

The denominator is positive and zero off-diagonal entries simply give a zero floor. Coefficient positivity and $(k+1)/2^{k/2} \ge 7/8$ for $0 \le k \le 6$ give $P \ge 7F(\sqrt{2})/8$. Squaring gives $P^2 \ge 396900 \prod_i Q_{ij}/Q_{jj}^3$; condition (61) makes this at least $10/3$. Equation (57) proves the compensation statement. If $Q_{ij} \ge 1/9$, a column sum gives $Q_{jj} \le 4/9$; thus the projection floor is at least $720(1/9)^3/(4/9)^{3/2} = 10/3$. The remaining constants follow from $(35/12)^2 - 10/3 = 745/144$ and $(10/3)/(35/12)^2 = 96/245$. □

Corollary 24.3 (Spectral and trace deficit domains)**.** *Let A_{count} count six independent categorical variables whose row probabilities are the rows of balanced Q. Define*

$$D = I - J_6/6 - \operatorname{Cov}(A_{\mathrm{count}}), \qquad W_{\mathrm{count}} = \operatorname{Tr} D/2.$$

Either $\lambda_{\max}(D) \le 1/225$ or $W_{\mathrm{count}} \le 1/450$ implies every $Q_{ij} \ge 1/9$, hence all the conclusions of Theorem 24.2. The spectral sufficient domain strictly contains the trace sufficient domain even on actual positive-definite Gram pencils.

Proof. Independence gives $\mathrm{Cov}(A_{\mathrm{count}}) = I - Q^{\mathsf{T}}Q$. For $T = Q - J_6/6$ and $\Pi = I - J_6/6$, double stochasticity yields $T = \Pi T \Pi$ and $D = T^{\mathsf{T}}T \succeq 0$. Therefore

$$|T_{ij}| \le \|\Pi e_i\| \|T\|_{\mathrm{op}} \|\Pi e_j\| = \tfrac{5}{6}\sqrt{\lambda_{\max}(D)}.$$

The spectral threshold gives $|T_{ij}| \le 1/18$ and thus $Q_{ij} \ge 1/9$. The trace threshold implies the spectral threshold by $D \succeq 0$.

For strictness take $Q = J_6/6 + \Pi/20$, whose diagonal is $5/24$ and whose off-diagonal entries are $19/120$. Then $D = \Pi/400$, so $\lambda_{\max}(D) = 1/400 < 1/225$, whereas $W_{\mathrm{count}} = 1/160 > 1/450$. It is actual Gram data: put $d_0 = (5/24)^{1/4}$, $b_0 = (19/120)^{1/4}$,

$$M = (d_0 - b_0)I + b_0 J_6, \quad H = (d_0 - b_0)\Pi, \quad X = \frac{d_0 + 5b_0}{6\sqrt{2}} J_6.$$

Then $H + \sqrt{2}X = M$, $|M|^4 = Q$, and $B = H + X$ is positive definite, with residual rank five and an active selected direction. Its common diagonal can be normalized to unit rows, after which balancing returns this same Q. Strictness concerns the covariance-statistic domains; the point already belongs to the direct entry and column domains. □

The auxiliary generating polynomial $\prod_i \sum_j Q_{ij} v_j$ is normalized homogeneous real stable: each row linear form has strictly positive imaginary part when all variables do. This is an actual independent categorical law. It does not make the ordinary occupation polynomial F stable, and no statistical-distance theorem is needed for the deterministic norm bound above. The CLL and scaling ingredients are classical; their application here provides explicit sufficient domains. Existing sufficient-domain work, including correlation-matrix annular conditions [20], remains relevant. The new column and deficit tests do not follow merely from $F(\sqrt{2}) > P$. The missing generic estimate is still the same-pencil payment

$$\Delta_{\sqrt{2}} + 2(\sqrt{2} - 1)E + R_{\sqrt{2}-1} \ge F(\sqrt{2})^2 - P^2$$

on the uncovered compact chart, including its rank-five and complex-phase boundaries. All displayed reserves and capacity tests remain sufficient conditions rather than a proof of that unrestricted statement.

25 Scope and reproducibility

The sharp theorem is analytic. The sector refinement uses classical Schur weights and content eigenvalues; its specialization and mixed-permanent evaluation produce the stated coupled bound. The moment relaxation and thinning algebra are classical tools. The cofactor PSD construction in Proposition 3.3 is credited to the existing omitted-tensor method; its role here is the residual coefficient cancellation.

The standalone standard-library program `verify.py` reconstructs the displayed factors, exact ranks, phase witnesses, mixed permanents, occupation coefficients, Newton defects and strict scalar margins. It uses rational arithmetic and explicit checks that remain active under `python -O`. No private computational files are needed.

The extensions use classical reciprocal-linear-form independence, bosonic creation calculus, numerical-range convexity and rigorous partition methods. Their role is the explicit variance geometry, relative-factor transfer and actual-quartic certificate. The finite adaptive example improves a two-leaf bound over the corresponding single full-box bound, while the exact shifted cofactor surrogate also passes that example. It is not a separation from that surrogate. The standalone `verify_adaptive.py` checks the supplied rational tree certificates, scalar constants and rejection tests. The standalone `verify_uniform.py` checks the restricted contraction norm, a genuinely mixed six-operator tensor fixture, exact permutation counts, rational reserve constants and the sharp quadratic degeneration. The analytic arguments, rather than those finite checks, establish the uniform statements. The trace-one extension uses the actual operator variance; its Bernoulli specialization is restricted to rank-one factors.

The order-five hypothesis in the earlier occupied-vertex transfer is discharged by Lemma 11.1; the public adaptive geometric checker conservatively retains its two-high-row rule.

The standalone `verify_coupling.py` corroborates the orthogonal trace sectors, support-identity kernel correction, frame-box conditions, secant constants and inherited-reserve arithmetic using exact examples and rejection controls. The universal theorems follow from the displayed proofs. The new compensation domains are sufficient conditions; their complements are not counterexamples and are not certified by sampling. The standalone `verify_forward.py` checks the new rational thresholds, actual Gram fixtures, finite-forward identity and defect-transfer obstruction. Finite checks corroborate the examples. The standalone `verify_occupation.py` reconstructs the new all-order product bound and occupation-resolved coefficients on exact finite examples, the named earlier-floor comparison and the kernel-ray identities. These checks support the normalizations and examples; the universal theorems follow from the displayed proofs. The conclusions in [7] concern Haar averages.

The standalone `verify_reserves.py` checks exact constants, cofactor occupation fixtures, curvature and projection normalizations, and the balanced-domain arithmetic. It corroborates finite identities and examples; the boundary band, overlap envelope and sharpness assertions follow from the analytic proofs in Sections 21–24. Classical CLL and matrix scaling are credited as inputs, rather than claimed as new results. The standalone `verify_root_product.py` checks the joint root-product constants and actual complex Gram fixtures, including active rank-six and singular rank-five examples. These finite checks corroborate normalizations and exact examples; Theorem 23.2 follows from its displayed analytic proof, which retains the weighted occupation budget and coherent product. Its $1/31$ domain reduces the positive-loss chart under consideration to $K_{ii} < 30$.

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